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Aims and Scope

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Contents

STRONGLY ALMOST SUMMABLE DIFFERENCE SEQUENCES AND STATISTICAL CONVERGENCE

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A B S T R A C T

The idea of di erence sequence space was introduced by Kizmaz [12] and was generalized by Et and Çolak [6]. In this paper, we introduce and examine some properties of three sequence spaces de ned by using a modulus function and give various properties and inclusion relation on these spaces.

1. INTRODUCTION

Let ω be the set of all sequences of real numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $||x|| = \sup|x_k|$, where $k \in \mathbb{N} = \{1, 2, 3.....\}$, the positive integers.

A sequence $x \in \ell_\infty$ is said to be almost convergent [14] if all Banach limits of x coincide. Lorentz [14] defined:

$$\hat{c} = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+m} \text{ exists, uniformly in } m \right\}.$$

Several authors including Lorentz [14], Duran [2] and King [11], have studied almost convergent sequences. Maddox [16, 17] has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \left| x_{k+m} - L \right| = 0 \text{ uniformly in } m.$$

By $[\hat{c}]$ we denote the spaces of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$.

The space of strongly almost convergent sequences was generalized by Nanda [20, 21].

Let $p = (p_k)$ be a sequence of strictly positive numbers. Nanda [20] defined:

$$[\hat{c}, p] = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \left| x_{k+m} - L \right|^{p_k} = 0, \text{ uniformly in } m \right\},$$

$$[\hat{c}, p]_0 = \left\{ x = (x_k) : \lim_n \frac{1}{n} \sum_{k=1}^{n} \left| x_{k+m} \right|^{p_k} = 0, \text{ uniformly in } m \right\},$$

$$[\hat{c}, p]_\infty = \left\{ x = (x_k) : \sup_{m,n} \frac{1}{n} \sum_{k=1}^{n} \left| x_{k+m} \right|^{p_k} < \infty \right\}.$$

Let $\lambda = (\lambda_k)$ be a nondecreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1,\ \lambda_1 = 1$.

The generalized de la Vallée-poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$

A sequence $x = (x_k)$ is said to be (V, λ) summable to a number L (see [13]), if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then (V, λ) summability and strongly (V, λ) summability are reduced to $(C, 1)$ summability and $[C, 1]$ summability, respectively.

The idea of difference sequence spaces was introduced by Kizmaz in [12]. In 1981, Kizmaz defined the sequence spaces:

$$X(\triangle) = \Big\{ x = (x_k) : \triangle x \in X \Big\}$$

for $X = \ell_\infty, c$ and c_0, where $\triangle x = \big(x_k - x_{k+1}\big)$.

Then Et and Çolak [6] generalized the above sequence spaces as below:

$$X(\triangle^r) = \Big\{ x = (x_k) : \triangle^r x \in X \Big\}$$

for $X = \ell_\infty, c$ and c_0, where $r \in \mathbb{N}$, $\triangle^0 x = (x_k)$, $\triangle x = (x_k - x_{k+1})$, $\triangle^r x = (\triangle^{r-1} x_k - \triangle^{r-1} x_{k+1})$, and so that

$$\triangle^r x = \sum_{v=0}^{r} (-1)^v \begin{bmatrix} r \\ v \end{bmatrix} x_{k+v}.$$

Recently, Et and Basarir [5] extended the above sequence spaces to the sequence spaces $X(\triangle^r)$ for $X = \ell_\infty(p)$, $c(p)$, $c_0(p)$, $\left[\hat{c}, p\right]$, $\left[\hat{c}, p\right]_0$ and $\left[\hat{c}, p\right]_\infty$.

We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \le f(x) + f(y)$ for all $x \ge 0,\ y \ge 0$,

(iii) f is increasing,

(iv) f is continuous from right at 0.

It follows that f must be a continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. Ruckle [23] and Maddox [15] used a modulus function f to construct some sequence spaces. Subsequently modulus function has been discussed in [3, 4, 19, 22, 26].

Further, let $X, Y \subset \omega$. Then we shall write ([27]):

$$M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \Big\{ a \in \omega : ax \in Y \ \text{ for all } x \in X \Big\}.$$

The set $X^{\alpha} = M(X, \ell_1)$ is called the Köthe-Toeplitz dual space or α- dual of X. Let X be a sequence space. Then X is called:

(i) Solid (or normal) if $(\alpha_k x_k) \in X$, whenever, $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.
(ii) Symmetric if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$ whenever $\pi(k)$ is a permutation of $\mathbb{N}$.
(iii) Perfect if $X = X^{\alpha\alpha}$.
(iv) Sequence algebra if $x.y \in X$, whenever $x, y \in X$. It is well known that if X is perfect then X is normal [10].

The following inequality will be used throughout this paper:

$$|a_k + b_k|^{p_k} \leq C\left[\ |a_k|^{p_k} + |b_k|^{p_k}\right], \tag{1.1}$$

where $a_k, b_k \in \mathbb{C}$, $0 \leq p_k \leq \sup_k p_k = H$, $C = \max\left(1,\ 2^{H-1}\right)$, (see [18]).

2. Main Results

In this section we prove some results involving the sequence space $[\hat{V}, \lambda, f, p]_0(\triangle^r, E)$, $[\hat{V}, \lambda, f, p]_1(\triangle^r, E)$ and $[\hat{V}, \lambda, f, p]_\infty(\triangle^r, E)$.

Definition 2.1. *Let E be Banach space. We define $\omega(E)$ to be the vector space of all E-valued sequences that is $\omega(E) = \left\{x = (x_k) : x_k \in E\right\}$. Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers.*

We define the following sequence sets:

$$[\hat{V}, \lambda, f, p]_1(\triangle^r, E) =$$

$$= \left\{x \in \omega(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[f\left(|\triangle^r x_{k+m} - L|\right)\right]^{p_k} = 0, \text{ uniformly in } m \text{ for some } L > 0\right\},$$

$$[\hat{V}, \lambda, f, p]_0(\triangle^r, E) = \left\{x \in \omega(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[f\left(|\triangle^r x_{k+m}|\right)\right]^{p_k} = 0, \text{uniformly in } m\right\},$$

and

$$[\hat{V}, \lambda, f, p]_\infty(\triangle^r, E) = \left\{x \in \omega(E) : \sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[f\left(|\triangle^r x_{k+m}|\right)\right]^{p_k} \leq \infty\right\}.$$

If $x \in [\hat{V}, \lambda, f, p]_1(\triangle^r, E)$ then we shall write $x_k \to L[\hat{V}, \lambda, f, p]_1(\triangle^r, E)$ and L will be called λ_E- strongly almost difference limit of x with respect to the modulus f.

Through this paper, Z will denote any one of the notions $0, 1$ or ∞.

In this case $f(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$, we shall write $[\hat{V}, \lambda]_z(\triangle^r, E)$ and $[\hat{V}, \lambda, f]_z(\triangle^r, E)$ instead of $[\hat{V}, \lambda, f, p]_z(\triangle^r, E)$. If $x \in [\hat{V}, \lambda]_1(\triangle^r, E)$ then we say that x is $\triangle^r_{\lambda, E}$ strongly almost convergent to L.

The proofs of the following two theorems are obtained by using the known standard techniques, therefore we give them without proofs (see for detail [3, 22]).

Theorem 2.1. *Let $p = (p_k)$ be a bounded. Then the spaces $[\hat{V}, \lambda, f, p]_z(\Delta^r, E)$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Theorem 2.2. *Let $p = (p_k)$ be a bounded and f be modulus function, then*

$$[\hat{V}, \lambda, f, p]_0(\Delta^r, E) \subset [\hat{V}, \lambda, f, p]_1(\Delta^r, E \subset [\hat{V}, \lambda, f, p]_\infty(\Delta^r, E).$$

Theorem 2.3. *If $r \geq 1$, then the inclusion $[\hat{V}, \lambda, f, p]_z(\Delta^{r-1}, E) \subset [\hat{V}, \lambda, f, p]_z(\Delta^r, E)$ is strict. In general $[\hat{V}, \lambda, f, p]_z(\Delta^i, E) \subset [\hat{V}, \lambda, f, p]_z(\Delta^r, E)$ for all $i = 1, 2, 3...r-1$ and the inclusion is strict.*

Proof. We give the proof for $Z = \infty$ only. It can be proved in similar way for $Z = 0, 1$. Let $x \in [\hat{V}, \lambda, f, p]_\infty(\Delta^{r-1}, E)$. Then we have:

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^{r-1} x_{k+m}\right|\right) < \infty.$$

By definition of f, we have:

$$\frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^r x_{k+m}\right|\right) \leq \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^{r-1} x_{k+m}\right|\right) + \frac{1}{\lambda_n} \sum_{k \in I_n} f\left(\left|\Delta^{r-1} x_{k+m+1}\right|\right) < \infty.$$

Thus,

$$[\hat{V}, \lambda, f, p]_\infty(\Delta^{r-1}, E) \subset [\hat{V}, \lambda, f, p]_\infty(\Delta^r, E).$$

Proceeding in this way, we have:

$$[\hat{V}, \lambda, f, p]_\infty(\Delta^i E) \subset [\hat{V}, \lambda, f, p]_\infty(\Delta^r, E),$$

for all $i = 1, 2, 3...r-1$. Let $\lambda_n = n$ for all $n \in \mathbb{N}$. Then the sequence $x = (k^r)$, for example, belongs to $[\hat{V}, \lambda, f, p]_\infty(\Delta^r, E)$, but doesn't belong to $[\hat{V}, \lambda, f, p]_\infty(\Delta^{r-1}, E)$ for $f(x) = x$ (if $x = (k^r)$, then $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1} x_k = (-1)^{r+1} r!(k + \frac{(r-1)}{2})$ for all $k \in \mathbb{N}$). □

Similarly, as in the previous theorems for the cases $[\hat{V}, \lambda, f, p]_0(\Delta^r, E)$ and $[\hat{V}, \lambda, f, p]_1(\Delta^r E)$ we have:

Proposition 2.1. *Let f be a sequence of modulus functions. Then:*

$$[\hat{V}, \lambda, f, p]_1(\Delta^{r-1}, E) \subset [\hat{V}, \lambda, f, p]_0(\Delta^r, E).$$

Theorem 2.4. *Let f_1 and f_2 be modulus functions. Then we have:*

(*i*) $[\hat{V}, \lambda, f_1, p]_z(\Delta^r, E) \subset [\hat{V}, \lambda, f_1 o f_2, p]_z(\Delta^r, E)$,
(*ii*) $[\hat{V}, \lambda, f_1, p]_z(\Delta^r, E) \cap [\hat{V}, \lambda, f_2, p]_z(\Delta^r, E) \subset [\hat{V}, \lambda, f_1 + f_2, p]_z(\Delta^r, E)$.

The following results are consequence of Theorem 2.4.

Proposition 2.2. *Let f be a modulus functions. Then:*

$$\left[\hat{V},\lambda,p\right]_z(\triangle^r,E)\subset\left[\hat{V},\lambda,f,p\right]_z(\triangle^r,E).$$

Theorem 2.5. *The sequence spaces $\left[\hat{V},\lambda,f,p\right]_0(\triangle^r,E)$, $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$ and $\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$ are not solid for $r\geq 1$.*

Proof. Let $p_k=1$ for all k, $f(x)=x$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $(x_k)=(k^r)\in[\hat{V},\lambda,f,p]_\infty(\triangle^r,E)$ but $(\alpha_k x_k)\notin[\hat{V},\lambda,f,p]_\infty(\triangle^r,E)$, when $\alpha_k=(-1)^k$ for all $k\in\mathbb{N}$. Hence $[\hat{V},\lambda,f,p]_\infty(\triangle^r,E)$ is not solid. The other cases can be proved by considering similar examples. □

From the above theorem we may give the following corollary.

Corollary 2.1. *The sequence spaces $\left[\hat{V},\lambda,f,p\right]_0(\triangle^r,E)$, $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$ and $\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$ are not perfect for $r\geq 1$.*

Theorem 2.6. *The sequence spaces $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$ and $\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$ are not symmetric for $r\geq 1$.*

Proof. Let $(p_k)=1$ for all k, $f(x)=x$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $x_k=(k^r)\in\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$. Let (y_k) be an arrangement of (x_k), which is defined by
$(y_k)=\left\{x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},x_8,x_{49},x_{10}\ldots\ldots\right\}$.

Then $(y_k)\notin\left[\hat{V},\lambda,f,p\right]_\infty(\triangle^r,E)$. □

Remark 2.1. *The space $[\hat{V},\lambda,f,p]_0(\triangle^r,E)$ is not symmetric for $r\geq 2$.*

Theorem 2.7. *The sequence spaces $\left[\hat{V},\lambda,f,p\right]_z(\triangle^r,E)$. are not sequence of algebras.*

Proof. Let $p_k=1$ for all $k\in\mathbb{N}.$, $f(x)=x$ and $\lambda_n=n$ for all $n\in\mathbb{N}$. Then $x=(k^{r-2})$, $y=(k^{r-2})\in\left[\hat{V},\lambda,f,p\right]_z(\triangle^r,E)$, but $x,y\in\left[\hat{V},\lambda,f,p\right]_z(\triangle^r,E)$. □

3. Statistical Convergent

The notion of statistical convergence was introduced by Fast [7] and studied by various authors [1, 9, 24, 25]. In this section we define $\triangle^r_{\lambda,E}$ almost statistically convergent sequences and give some inclusion relations between $\hat{s}(\triangle^r_\lambda,E)$ and $\left[\hat{V},\lambda,f,p\right]_1(\triangle^r,E)$.

Definition 3.1. *A sequence $x=(x_k)$ is said to be $\triangle^r_{\lambda,E}$-almost statistically convergent to the number L if for every $\epsilon>0$,*

$$\lim_n\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\triangle^r x_{k+m}-L\right|\geq\epsilon\right\}\right|=0 \text{ uniformly in } m.$$

In this case we write $\hat{s}(\triangle_\lambda^r, E) - \lim x = L$, or $x_k \to L\hat{s}(\triangle_\lambda^r, E)$.

When $\lambda_n = n$ and $L = 0$ we shall write $\hat{s}(\triangle^r, E)$ instead of $\hat{s}(\triangle_\lambda^r, E)$.

The proof of the following theorem is easily obtained by using the same technique as in Theorem 2 in Savaş [25], therefore we give it without proof.

Theorem 3.1. *Let $\lambda = (\lambda_n)$ be the same as in section 1, then:*

(i) *If $x_k \to L[\hat{V}, \lambda]_1(\triangle^r, E) \Rightarrow x_k$ then $L\hat{s}(\triangle_\lambda^r, E)$;*

(ii) *If $x \in \ell_\infty(\triangle^r, E)$ and $x_k \to L\hat{s}(\triangle_\lambda^r, E)$, then $x_k \to L[\hat{V}, \lambda]_1(\triangle^r, E)$;*

(iii) $\hat{s}(\triangle_\lambda^r, E) \cap \ell_\infty(\triangle^r, E) = [\hat{V}, \lambda]_1(\triangle^r, E) \cap \ell_\infty(\triangle^r, E)$.

Theorem 3.2. $\hat{s}(\triangle^r, E) \subset \hat{s}(\triangle_\lambda^r, E)$ *if and only if* $\liminf_n \frac{\lambda_n}{n} > 0$.

Proof. The sufficiency part of this proof can be obtained using the same technique as the sufficiency part of proof of Theorem 3 in Savaş [25].

For necessity suppose that $\liminf_n \frac{\lambda_n}{n} = 0$. As in ([8], p. 510) we can choose a subsequence $(n(j))$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$. We define $x = (x_i)$ such that:

$$\triangle^r x_i = \begin{cases} 1 \text{ if } i \in I_n(j),\ j = 1, 2, 3.....; \\ 0 \text{ otherwise.} \end{cases}$$

Then $x \in [\hat{c}](\triangle^r, E)$ and by [4, Theorem 3.1 (i)], $x \in \hat{s}(\triangle^r, E)$. But $x \notin [\hat{V}, \lambda]_1(\triangle^r, E)$ and Theorem 3.1 (ii) implies that $x \notin \hat{s}(\triangle_\lambda^r, E)$. This completes the proof. □

Theorem 3.3. *Let f be a modulus function and* $\sup_k p_k = H$. *Then:*

$$[\hat{V}, \lambda, f, p]_1(\triangle^r, E) \subset \hat{s}(\triangle_\lambda^r, E).$$

Proof. Let $x \in [\hat{V}, \lambda, f, p]_1(\triangle^r, E)$ and $\epsilon > 0$ be given. Let Σ_1 denote the sum over $k \leq n$ such that $|\triangle^r x_{k+m} - L| \geq \epsilon$ and Σ_2 denote the sum of over $k \leq n$ such that $|\triangle^r x_{k+m} - L| < \epsilon$. Then:

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[f(|\triangle_{k+m}^r - L|) \right]^{p_k} &= \frac{1}{\lambda_n} \sum_1 \left[f(|\triangle_{k+m}^r - L|) \right]^{p_k} + \frac{1}{\lambda_n} \sum_2 \left[f(|\triangle_{k+m}^r - L|) \right]^{p_k} \\ &\geq \frac{1}{\lambda_n} \sum_1 \left[f(|\triangle_{k+m}^r - L|) \right]^{p_k} \geq \frac{1}{\lambda_n} \sum_1 [f(\epsilon)]^{p_k} \\ &\geq \frac{1}{\lambda_n} \sum_1 \min\left([f(\epsilon)]^{\inf p_k}, [f(\epsilon)]^H \right) \\ &\geq |\{k \in I_n : |\triangle^r x_{k+m} - L| \geq \epsilon\}| \min\left([f(\epsilon)]^{\inf p_k}, [f(\epsilon)]^H \right) \end{aligned}$$

Hence $x \in \hat{s}(\triangle_\lambda^r, E)$. $\square$

Theorem 3.4. *Let f be a bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then: $\hat{s}(\triangle_\lambda^r, E) \subset \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E)$.*

Proof. Suppose that f is bounded. Let $\epsilon > 0$ and Σ_1 and Σ_2 be denoted in the previous theorem. Since f is bounded there exists an integer K such that $f(x) < K$ for all $x \geq 0$. Then:

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left|\triangle_{k+m}^r - L\right|\right)\right]^{p_k} &= \frac{1}{\lambda_n}\sum_1\left[f\left(\left|\triangle_{k+m}^r - L\right|\right)\right]^{p_k} + \frac{1}{\lambda_n}\sum_2\left[f\left(\left|\triangle_{k+m}^r - L\right|\right)\right]^{p_k}\\ &\leq \frac{1}{\lambda_n}\sum_1 \max\left(K^h, K^H\right) + \frac{1}{\lambda_n}\sum_2\left[f(\epsilon)\right]^{p_k}\\ &\leq \max\left(K^h, K^H\right)\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\triangle_{k+m}^r - L\right| \geq \epsilon\right\}\right| +\\ &+ \max\left(f(\epsilon)^h, f(\epsilon)^H\right)\end{aligned}$$

Hence $x \in \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E)$. $\square$

Theorem 3.5. *Let $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then*

$$\hat{s}(\triangle_\lambda^r, E) = \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E),$$

if and only if f is bounded.

Proof. Let f is bounded bounded. By Theorem 3.4 and Theorem 3.5 we have $\hat{s}\left(\triangle_\lambda^r, E\right) = \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E)$. Conversely, suppose that f is unbounded. Then there exists a positive sequence (t_k) with $f(t_k) = k^2$, for $k = 1, 2, 3.....$ If we choose

$$\triangle^r x_i = \begin{cases} t_k \ i = k^2, \ i = 1, 2, 3...... \\ 0 \text{ otherwise.}\end{cases},$$

then we have:

$$\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\triangle_{k+m}^r\right| \geq \epsilon\right\}\right| \leq \frac{\sqrt{\lambda_{n-1}}}{\lambda_n} \text{ for all } n, m$$

and so that $x \in \hat{s}(\triangle_\lambda^r, E)$ but $x \notin \left[\hat{V}, \lambda, f, p\right]_1(\triangle^r, E)$ for $E = \mathbb{C}$. This contradict to $\hat{s}\left(\triangle_\lambda^r, E\right) = \left[\hat{V}, \lambda, f, p\right]_1(\triangle_\lambda^r, E)$ $\square$

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ON A PRODUCT SUMMABILITY OF AN ORTHOGONAL SERIES

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A B S T R A C T

In this paper we have de ned a new product summability, in order to make an advanced study in the special topic of summability. Namely, we give some su cient conditions, in terms of the coe cients of an orthogonal series, under which such series is product summable almost everywhere.

1. INTRODUCTION

The absolute summability is a generalization of the concept of the absolute just as the summability is an extension of the concept of the convergence. There are a lot of notions of absolute summability de ned by several authors. Particularly, by those authors such notions are employed for studying the absolute summability of an series. As a recent result can be mentioned those of Y. Okuyama (see section 2) who has proved two theorems which give su cient conditions in terms of the coe cients of an orthogonal series under which such series would be absolute generalized Nörlund summable almost everywhere. Moreover, an interested reader could nd some new results, see as examples [4]-[6], where are given some statements which include all of the results previously proved by Y. Okuyama and T. Tsuchikura [8]-[9], and also are given some new consequences. In order to make an advance study in this direction, here we study the question when an orthogonal series is product summable almost everywhere.

2. Notations and Known Results

For two sequences of real or complex numbers $\{p_n\}$ and $\{q_n\}$, let

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n = \sum_{v=0}^{n} p_v,$$

$$Q_n = q_0 + q_1 + q_2 + \cdots + q_n = \sum_{v=0}^{n} q_v,$$

and let the convolution $(p * q)_n$ be defined by

$$R_n := (p * q)_n := \sum_{v=0}^{n} p_v q_{n-v}, \quad \text{and} \quad \text{denote} \quad R_n^j := \sum_{v=j}^{n} p_v q_{n-v}.$$

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of its $n-$th partial sums $\{s_n\}$. We write

$$t_n^{p,q} = \frac{1}{R_n} \sum_{\nu=0}^{n} p_{n-\nu} q_\nu s_\nu.$$

If $R_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable (N, p, q) if the series

$$\sum_{n=1}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|.$$

The $|N, p, q|$ summability was introduced by Tanaka [3].

Let $\{\varphi_j(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that f belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{j=0}^{\infty} c_j \varphi_j(x), \tag{2.1}$$

where $c_j = \int_a^b f(x)\varphi_j(x)dx$, $(j = 0, 1, 2, \ldots)$.

Regarding to the orthogonal series (2.1) Y. Okuyama has proved the following two theorems:

Theorem 2.1 ([8]). *If the series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j \varphi_j(x)$$

is summable $|N, p, q|$ almost everywhere.

Theorem 2.2 ([8]). *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w(n)$$

converges, then the orthogonal series

$$\sum_{j=0}^{\infty} c_j \varphi_j(x) \in |N, p, q|$$

almost everywhere, where $w(n)$ is defined by

$$w(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 .$$

If we take $p_v = 1$ for all v then, the sequence-to-sequence transformation $t_n^{p,q}$ reduces to transformation $R_n^q := \frac{1}{Q_n} \sum_{v=0}^{n} q_v s_v$, while for $q_v = 1$ we obtain the transformation $R_n^p := \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v$. G. Das [2] defined the transformation

$$U_n := \frac{1}{P_n} \sum_{v=0}^{n} \frac{p_{n-v}}{Q_v} \sum_{j=0}^{v} q_{v-j} s_j ,$$

and gave the following definition:

Definition 2.1. *The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|(N,p)(N,q)|$, if the sequence $\{U_n\}$ is of bonded variation, i.e. the series*

$$\sum_{n=1}^{\infty} |U_n - U_{n-1}|$$

converges.

Later on, W. T. Sulaiman [1] considered the transformation

$$V_n := \frac{1}{Q_n} \sum_{v=0}^{n} \frac{q_v}{P_v} \sum_{j=0}^{v} p_j s_j$$

of the sequence $\{s_n\}$, and presented the definition:

Definition 2.2. *The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|(R, q_n)(R, p_n)|_k$, $k \geq 1$, if*

$$\sum_{n=1}^{\infty} n^{k-1} |V_n - V_{n-1}|^k$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |(R, q_n)(R, p_n)|_k.$$

Let us denote by D_n the transformation

$$D_n := \frac{1}{R_n} \sum_{v=0}^{n} \frac{p_{n-v} q_v}{R_v} \sum_{j=0}^{v} p_j q_{v-j} s_j$$

of the sequence $\{s_n\}$.

Now we shall introduce the following definition:

Definition 2.3. *The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|(N, p_n, q_n)(N, q_n, p_n)|_k$, $k \geq 1$, if*

$$\sum_{n=1}^{\infty} n^{k-1} |D_n - D_{n-1}|^k$$

converges, and we write shortly $\sum_{n=0}^{\infty} a_n \in |(N, p_n, q_n)(N, q_n, p_n)|_k$.

The main purpose of the present paper is to study the $|(N, p_n, q_n)(N, q_n, p_n)|_k$ summability of the orthogonal series (2.1) for $1 \leq k \leq 2$.

Throughout K denotes a positive constant that it may depends only on k, and be different in different relations.

The following lemma due to Beppo Levi (see, for example [7]) is often used in the theory of functions. It will need us to prove main results.

Lemma 2.1. *If $f_n(t) \in L(E)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_E f_n(t) dt < \infty, \tag{2.2}$$

then the series

$$\sum_{n=1}^{\infty} f_n(t)$$

converges almost everywhere on E to a function $f(t) \in L(E)$. Moreover, the series (2.2) is also convergent to f in the norm of $L(E)$.

3. Main Results

We prove the following theorem.

Theorem 3.1. *If for $1 \leq k \leq 2$ the series*

$$\sum_{n=1}^{\infty} \left[n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left(\frac{R_n^i \widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i \widetilde{R}_{n-1}^i}{R_{n-1}} \right)^2 |c_i|^2 \right]^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|(N,p_n,q_n)(N,q_n,p_n)|_k$ *almost everywhere.*

Proof. First we consider the case $k \in (1,2)$. We use the notations

$$\widetilde{R}_n^i := \sum_{v=i}^{n} \frac{q_{n-v}p_v}{R_v}; \quad \widetilde{R}_{n-1}^n = 0.$$

Let

$$s_j(x) = \sum_{i=0}^{j} c_i\varphi_i(x)$$

be the partial sums of order j of the series (2.1). Then, for the transform $D_n(x)$ of the partial sums $s_j(x)$, we have

$$\begin{aligned}
D_n(x) &= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{j=0}^{v}p_jq_{v-j}\sum_{i=0}^{j}c_i\varphi_i(x)\\
&= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{i=0}^{v}c_i\varphi_i(x)\sum_{j=i}^{v}p_jq_{v-j}\\
&= \frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{R_v}\sum_{i=0}^{v}R_v^ic_i\varphi_i(x)\\
&= \frac{1}{R_n}\sum_{i=0}^{n}R_n^ic_i\varphi_i(x)\sum_{v=i}^{n}\frac{p_{n-v}q_v}{R_v}\\
&= \sum_{i=0}^{n}\frac{R_n^i\widetilde{R}_n^i}{R_n}c_i\varphi_i(x).
\end{aligned}$$

Whence,

$$\begin{aligned}
\triangle D_n(x) &= D_n(x) - D_{n-1}(x)\\
&= \sum_{i=0}^{n}\frac{R_n^i\widetilde{R}_n^i}{R_n}c_i\varphi_i(x) - \sum_{i=0}^{n-1}\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}c_i\varphi_i(x)\\
&= \sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)c_i\varphi_i(x).
\end{aligned}$$

Using the Hölder's inequality and orthogonality to the latter equality, we obtain

$$\begin{aligned}
\int_a^b |\triangle D_n(x)|^k dx &\leq (b-a)^{1-\frac{k}{2}}\left(\int_a^b |D_n(x) - D_{n-1}(x)|^2 dx\right)^{\frac{k}{2}}\\
&= (b-a)^{1-\frac{k}{2}}\left[\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n} - \frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2 |c_i|^2\right]^{\frac{k}{2}}.
\end{aligned}$$

Subsequently, the series

$$(3.1)\quad \sum_{n=1}^{\infty} n^{k-1}\int_a^b |\triangle D_n(x)|^k dx \le K\sum_{n=1}^{\infty} n^{k-1}\left[\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2|c_i|^2\right]^{\frac{k}{2}}$$

converges, since the last one does. Now according to the Lemma 2.1 the series (2.1) is summable $|(N,p_n,q_n)(N,q_n,p_n)|_k$ almost everywhere. For $k=2$ we apply only the

orthogonality, as far as for $k=1$ we apply the well-known Schwarz's inequality. This completes the proof of the theorem. □

Now we shall prove the counterpart of Theorem 3.1 (it can be seen also as the counterpart of a theorem of P. L. Ul'yanov [10]). It is a general theorem which involves in it a new positive sequence with some additional conditions. For this reason first we put:

$$(3.2)\qquad \mathfrak{R}^{(k)}(i) := \frac{1}{i^{\frac{2}{k}-1}}\sum_{n=i}^{\infty} n^{\frac{2}{k}}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2,$$

and then the following theorem holds true.

Theorem 3.2. *Let $1\le k\le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges.*

Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series

$$\sum_{n=1}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\mathfrak{R}^{(k)}(n)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)\in|(N,p_n,q_n)(N,q_n,p_n)|_k$$

almost everywhere, where $\mathfrak{R}^{(k)}(n)$ is defined by (3.2).

Proof. Applying Hölder's inequality to the inequality (3.1) we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{k-1}\int_a^b |\triangle D_n(x)|^k dx\\
&\le K\sum_{n=1}^{\infty} n^{k-1}\left[\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2|c_i|^2\right]^{\frac{k}{2}}\\
&= K\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))^{\frac{2-k}{2}}}\left[(n\Omega(n))^{\frac{2}{k}-1}n^{2-\frac{2}{k}}\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2|c_i|^2\right]^{\frac{k}{2}}
\end{aligned}$$

$$\leq K\left(\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}\right)^{\frac{2-k}{2}}\left[\sum_{n=1}^{\infty}(n\Omega(n))^{\frac{2}{k}-1}n^{2-\frac{2}{k}}\sum_{i=1}^{n}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2|c_i|^2\right]^{\frac{k}{2}}$$

$$\leq K\left\{\sum_{i=1}^{\infty}|c_i|^2\sum_{n=i}^{\infty}(n\Omega(n))^{\frac{2}{k}-1}n^{2-\frac{2}{k}}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2\right\}^{\frac{k}{2}}$$

$$\leq K\left\{\sum_{i=1}^{\infty}|c_i|^2\left(\frac{\Omega(i)}{i}\right)^{\frac{2}{k}-1}\sum_{n=i}^{\infty}n^{\frac{2}{k}}\left(\frac{R_n^i\widetilde{R}_n^i}{R_n}-\frac{R_{n-1}^i\widetilde{R}_{n-1}^i}{R_{n-1}}\right)^2\right\}^{\frac{k}{2}}$$

$$\leq K\left\{\sum_{i=1}^{\infty}|c_i|^2\Omega^{\frac{2}{k}-1}(i)\mathfrak{R}^{(k)}(i)\right\}^{\frac{k}{2}},$$

which is finite by assumption. Doing the same reasoning as in the proof of Theorem 3.1 we easy arrive to finish the proof. □

It is obvious that the transformation D_n can never be the same as $\sigma_n^{p,q}$, therefore Theorems 3.1-3.2 bring new results. Moreover, transformation D_n can not reduces to the transformations U_n or V_n, i.e. it would be of particular interest to answer questions: Under what conditions an orthogonal series of the form (2.1) is $(N,p)(N,q)$ or $(R,q_n)(R,p_n)_k$ summable? Regarding to these questions, in the following, we shall give four theorems without their proofs.

Denote

$$\widetilde{P}_n^{i,q}=\sum_{v=i}^{n}\frac{p_{n-v}}{Q_v},\quad \widetilde{Q}_n^{i,p}=\sum_{v=i}^{n}\frac{q_v}{P_v},\quad \text{and}\quad \widetilde{P}_{n-1}^{n,q}=\widetilde{Q}_{n-1}^{n,p}=0.$$

Theorem 3.3. *If the series*

$$\sum_{n=1}^{\infty}\left[\sum_{i=1}^{n}\left(\frac{Q_n^i\widetilde{P}_n^{i,q}}{P_n}-\frac{Q_{n-1}^i\widetilde{P}_{n-1}^{i,q}}{P_{n-1}}\right)^2|c_i|^2\right]^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)$$

is summable $|(N,p)(N,q)|$ *almost everywhere.*

Theorem 3.4. *Let* $\{\Omega(n)\}$ *be a positive sequence such that* $\{\Omega(n)/n\}$ *is a non-increasing sequence and the series* $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ *converges.*

Let $\{p_n\}$ *and* $\{q_n\}$ *be non-negative. If the series*

$$\sum_{n=1}^{\infty}|c_n|^2\Omega(n)\mathfrak{R}^{p,q}(n)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |(N,p)(N,q)|$$

almost everywhere, where $\Re^{p,q}(n)$ is defined by

$$\Re^{p,q}(i) := \frac{1}{i} \sum_{n=i}^{\infty} n^2 \left(\frac{Q_n^i \tilde{P}_n^{i,q}}{P_n} - \frac{Q_{n-1}^i \tilde{P}_{n-1}^{i,q}}{P_{n-1}} \right)^2 .$$

Theorem 3.5. *If for $1 \le k \le 2$ the series*

$$\sum_{n=1}^{\infty} \left[n^{2-\frac{2}{k}} \sum_{i=1}^{n} \left(\frac{P_n^i \tilde{Q}_n^{i,p}}{Q_n} - \frac{P_{n-1}^i \tilde{Q}_{n-1}^{i,p}}{Q_{n-1}} \right)^2 |c_i|^2 \right]^{\frac{k}{2}}$$

converges, then the orthogonal series

is summable $|(R,q_n)(R,p_n)|_k$ almost everywhere.

Theorem 3.6. *Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges.*

Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) \hat{\Re}^{(p,q;k)}(n)$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |(R,q_n)(R,p_n)|_k$$

almost everywhere, where $\hat{\Re}^{(p,q;k)}(n)$ is defined by

$$\hat{\Re}^{(p,q;k)}(i) := \frac{1}{i^{\frac{2}{k}-1}} \sum_{n=i}^{\infty} n^{\frac{2}{k}} \left(\frac{P_n^i \tilde{Q}_n^{i,p}}{Q_n} - \frac{P_{n-1}^i \tilde{Q}_{n-1}^{i,p}}{Q_{n-1}} \right)^2 .$$

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DETERMINATION OF JUMPS OF A FUNCTION OF VP CLASS BY ITS INTEGRATED FOURIER-JACOBI SERIES

SAMRA PIRI,

ABSTRACT

Theproblemofdetermination of jump discontinuities in piecewise smooth functions from their spectral data is relevant in signal processing. We obtain new identity which determines the jumps of a periodic function of $\mathcal{V}_p, 1 \leqslant p < 2,$ *class with a nite number of discontinuities, by means of the tails of its integrated Fourier Jacobi series. Next, we establish* $(C,\alpha),\ \alpha > 1 - \frac{1}{p},$ *summability of the sequence* $(n^2 a_n(w; f) \int P_n(w; x) dx)$, where $a_n(w; f) \int$ *Pn (w;x)dx is the n-th term of the tegrated Fourier-Jacobi series of a function f.*

1. INTRODUCTION

The problem of locating the discontinuities of a function by means of its truncated Fourier series, arises naturally from an attempt to overcome the Gibbs phenomenon, poor approximative properties of the Fourier partial sums of a discontinuous function (i.e. the nite sum approximation of the discontinuous function overshoots the function itself, at a discontinuity by about 18 percent).

If a function f is integrable on $[-\pi, \pi]$, then it has a Fourier series with respect to the trigonometric system $\{1, cos\, nx, sin\, nx\}_{n=1}^{\infty}$, and we denote the n-th partial sum of the Fourier series of f by $S_n(x, f)$, i.e.,

$$S_n(x, f) = \frac{a_0(f)}{2} + \sum_{k=1}^{n} (a_k(f) cos\, kx + b_k(f) sin\, kx),$$

where $a_k(f) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) cos\, kt\, dt$ and $b_k(f) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) sin\, kt\, dt$ are the $k-$th Fourier coefficients of the function f.

The identity determining the jumps of a function of bounded variation by means of its differentiated Fourier partial sums has been known for a long time. Let $f(x)$ be a

function of bounded variation with period 2π, and $S_n(x, f)$ be the partial sum of order n of its Fourier series. By the classical theorem of Fejer [16] the identity:

$$\lim_{n\to\infty} \frac{S_n'(x, f)}{n} = \frac{1}{\pi}(f(x+0) - f(x-0)), \tag{1.1}$$

holds at any point x. To characterize continuous periodic functions of BV in terms of their Fourier coefficients, Wiener [15] has introduced a concept of higher variation.

A function f is said to be of bounded p-variation, $p \geq 1$, on the segment $[a, b]$ and to belong to the class $\mathcal{V}_p[a, b]$ if

$$V_a^b{}_p(f) = \sup_{\Pi_{a,b}} \left\{ \sum_i |f(x_i) - f(x_{i-1})|^p \right\}^{\frac{1}{p}} < \infty,$$

where $\Pi_{a,b} = \{a = x_0 < x_1 < ... < x_n = b\}$ is an arbitrary partition of the segment $[a, b]$. $V_a^b{}_p(f)$ is the p-variation of f on $[a, b]$.

B.I. Golubov [8] has shown that identity (1.1) is valid for classes $\mathcal{V}_p$.

Theorem (A). *Let $f(x) \in \mathcal{V}_p$, $(1 \leq p < \infty)$ and $r \in \mathbb{N}_0$. Then for any point x one has the equation*

$$\lim_{n\to\infty} \frac{S_n^{(2r+1)}(x, f)}{n^{2r+1}} = \frac{(-1)^r}{(2r+1)\pi}(f(x+0) - f(x-0)). \tag{1.2}$$

Problems of everywhere convergence of Fourier series for every change of variable have led D. Waterman [14] to another type of generalization.

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of positive numbers such that $\sum \frac{1}{\lambda_n}$ diverges and $\{I_n\}$ be a sequence of non overlapping segments $I_n = [a_n, b_n] \subset [a, b]$. A function f is said to be of Λ-bounded variation on $I = [a, b]$ $(f \in \Lambda BV)$ if $\sum \frac{|f(b_n) - f(a_n)|}{\lambda_n} < \infty$ for every choice of $\{I_n\}$. The supremum of these sums is called the Λ-variation of f on I. In the case $\Lambda = \{n\}$, one speaks of harmonic bounded variation (HBV).

The class HBV contains all Wiener classes. Avdispahic has shown in [3] that HBV is the limiting case for validity of the identity (1.1):

Theorem (B). *The equation (1.1) holds for any function $f \in HBV$ at any point x.*

The third interesting generalization of the Jordan variation was given by Z. A. Chanturiya [5]. The modulus of variation of a bounded 2π periodic function f is the function ν_f with domain the positive integers, given by

$$\nu_f(n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(b_k) - f(a_k)|,$$

where $\Pi_n = \{[a_k, b_k]; k = 1, ..., n\}$ is an arbitrary partition of $[0, 2\pi]$ into n non overlapping segments.

By a theorem of Avdispahic [1], there exist the following inclusion relations between Wiener's, Waterman's and Chanturiya's classes:

Theorem (C).
$$\{n^{\alpha}\}BV \subset V_{\frac{1}{1-\alpha}} \subset V[n^{\alpha}] \subset \{n^{\beta}\}BV,$$
for $0 < \alpha < \beta < 1$.

Clearly, Fejer's identity (1.1) is a statement about Cesaro summability of the sequence $\{kb_k \cos kx - ka_k \sin kx\}$, $a_k = a_k(f)$ and $b_k = b_k(f)$ being the k-th cosine and sine coefficient, respectively. Looking at Fejer's theorem in this way, several mathematicians have extended it to more general summability methods. We note two results [2] which represent the extension to (C, α) summability, $\alpha > 0$:

Theorem (D). *If* $f \in V_p$, $p > 1$ *the sequence* $\{kb_k \cos kx - ka_k \sin kx\}$ *is* (C, α) *summable to* $\frac{1}{\pi}(f(x+0) - f(x-0))$ *for any* $\alpha > 1 - \frac{1}{p}$ *and every* x.

Corollary(E). *If* $f \in V[n^{\beta}]$ $(\{n^{\beta}\}BV)$ *for some* $0 < \beta < 1$, *then the sequence* $\{kb_k \cos kx - ka_k \sin kx\}$ *is summable to* $\frac{1}{\pi}(f(x+0) - f(x-0))$ *by any Cesaro method of order* $\alpha > \beta$.

Theorem (D) and Corollary (E), are in some sense the most natural generalization of Fejer's theorem. Indicating the relationship between the order of Cesaro summability of the sequence $\{kb_k(f) \cos kx - ka_k(f) \sin kx\}$ and the "order of variation" of a function f, they complete the earlier picture whose elements were:

1) (C, α) summability for $\alpha > 0$ and the class BV;
2) (C, α) summability for $\alpha > 1$ and whole class of regulated functions (i.e. functions possessing the one-sided limits at each point);
3) $(C, 1)$ summability for the class HBV.

Similar identities hold if we consider the integrated rather than the differentiated Fourier series [9]. By $R_n(x, f)$ we denote the n-th order tails of the Fourier series of the function f, i.e.,
$$R_n(x, f) = \sum_{k=n}^{\infty} (a_k(f) coskx + b_k(f) sinkx),$$
for $n \in \mathbb{N}$.

For any function f, integrable on $[-\pi, \pi]$, $f^{(-r)}$, $r \in \mathbb{N}_0$, is defined as follows
$$f^{(-r-1)} \equiv \int f^{(-r)},$$
where $f^{(0)} \equiv f$, and the constants of integration are successively determined by the condition
$$\int_{-\pi}^{\pi} f^{(-r)}(t)dt = 0.$$

Theorem (F). *Let* $r \in \mathbb{N}_0$ *and suppose the function* $f \in V_p$, $1 \leqslant p < 2$, *has a finite number of discontinuities. Then:*

1. the identity

$$\lim_{n\to\infty} n^{2r+1} R_n^{(-2r-1)}(f;x) = \frac{(-1)^{r+1}}{(2r+1)\pi}(f(x+)-f(x-))$$

is valid for each fixed $x \in [-\pi,\pi]$;
2. there is no way to determine the jump at the point $x \in [-\pi,\pi]$ of an arbitrary function $f \in V_p$, $p \geqslant 1$, by means of the sequence $(R_n^{(-2r-2)}(f;.)), n \in \mathbb{N}$.

Such results find their application in recovering edges in piecewise smooth functions with finitely many jump discontinuities [6].

We say that a function w is a generalized Jacobi weight and write $w \in GJ$, if

$$w(t) = h(t)(1-t)^{\alpha}(1+t)^{\beta}|t-x_1|^{\delta_1}...|t-x_M|^{\delta_M},$$

$$h \in C[-1,1],\ h(t) > 0\ (|t| \leq 1),\ \omega(h;t;[-1,1])t^{-1} \in L[0,1],$$

$$-1 < x_1 < ... < x_M < 1,\ \ \alpha,\beta,\delta_1,...,\delta_M > -1.$$

By $\sigma(w) = (P_n(w;x))_{n=0}^{\infty}$ we denote the system of algebraic polynomials
$P_n(w;x) = \gamma(w)x^n +$ lower degree terms with positive leading coefficients $\gamma_n(w)$, which are orthonormal on $[-1,1]$ with respect to the weight $w \in GJ$, i.e.,

$$\int_{-1}^{1} P_n(w;t)P_m(w;t)w(t)dt = \delta_{nm}.$$

Such polynomials are called the generalized Jacobi polynomials. If $fw \in L[-1,1]$, and $w \in GJ$, then the n th partial sum of the Fourier series of f with respect to the system $\sigma(w)$ is given by

$$S_n(w;f;x) = \sum_{k=0}^{n-1} a_k(w;f)P_k(w;x) = \int_{-1}^{1} f(t)K_n(w;x;t)w(t)dt,$$

where $a_k(w;f) = \int_{-1}^{1} f(t)P_k(w;t)w(t)dt$ is the k th Fourier coefficient of the function f, and

$$K_n(w;x;t) = \sum_{k=0}^{n-1} P_k(w;x)P_k(w;t),$$

is the Dirichlet kernel of the system $\sigma(w)$.
For a given weight $w \in GJ$ it is assumed that $x_0 = -1$ and $x_{M+1} = 1$. In addition,

$$\Delta(\nu;\varepsilon) = [x_\nu + \varepsilon; x_{\nu+1} - \varepsilon],$$

for a fixed $\varepsilon \in (0, \frac{x_{\nu+1}-x_\nu}{2})$, $\ \nu = 1,2,...,M$.

For functions of $\Lambda-$bounded variation G. Kvernadze [10] has proved the following theorem:

Theorem (G). *Let $r \in \mathbb{N}_0$, $w \in GJ$, and suppose ΛBV is the class of functions of Λ-bounded variation determined by the sequence $\Lambda = (\lambda_k)_{k=1}^{\infty}$. Then the identity*

$$\lim_{n\to\infty} \frac{(S_n(w;f;x))^{2r+1}}{n^{2r+1}} = \frac{(-1)^r(1-x^2)^{-r-\frac{1}{2}}}{(2r+1)\pi}(f(x+0)-f(x-0)) \tag{1.3}$$

is valid for every $f \in \Lambda BV$ *and each fixed* $x \in (-1,1)$, $x \neq x_1, ..., x_M$, *if* $\Lambda BV \subseteq HBV$. *If, in addition, the weight* $w \in GJ$ *satisfies the following conditions:*

$$\alpha \geq -\frac{1}{2},\ \ \beta \geq -\frac{1}{2},\ \delta_1 \geq 0, ..., \delta_M \geq 0,\ \ \omega(h;t)t^{-1}\ln t \in L[0,1], \tag{1.4}$$

then condition $\Lambda BV \subseteq HBV$ *is necessary for the validity of identity (1.3) for every* $f \in \Lambda BV$ *and each fixed* $x \in (-1,1)$, $x \neq x_1, ..., x_M$ *as well.*

In [11], [12] is shown that the jump of a function f belonging to the Wiener class V_p, $p > 1$, can be determined through (C,α), $\alpha > 1 - \frac{1}{p}$, summability of the sequence of terms of it's differentiated Fourier-Jacobi series. Consesequently, the corresponding (C,α) summability result holds for the Waterman classes $\{n^\beta\}BV$ and the Chanturiya classes $V[n^\beta]$ if $\alpha > \beta$, $0 < \beta < 1$.

2. Main Results

Theorem 2.1. *Let* $r \in \mathbb{N}_0$ *and suppose the function* $f \in V_p$, $1 \leqslant p < 2$, *has a finite number of discontinuities and* $fw \in L[-1,1], w \in GJ$. *Then the identity*

$$\lim_{n\to\infty} nR_n^{(-1)}(w;f;x) = -\frac{1}{\pi}(1-x^2)^{\frac{1}{2}}(f(x+)-f(x-))$$

is valid for each fixed $x \in [-1,1]$, *where* $R_n^{(-1)}(w;f;x)$ *is the* n*-th order tails of the integrated Fourier-Jacobi series of the function* f.

Proof. By $S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)$ we denote the n-th partial sum of the Fourier-Tchebycheff series of function f [4]. We use the uniform equiconvergence of Fourier-Tchebycheff series and Fourier series with respect to the system of generalized Jacobi polynomials for an arbitrary function $f \in HBV$ and a fixed $\varepsilon \in (0, \frac{x_{\nu+1}-x_\nu}{2})$, $\nu = 0,1,2,...,M$

$$||S_n(w;f;x) - S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)||_{C[\Delta(\nu;\frac{\varepsilon}{2})]} = o(1), \tag{2.1}$$

proved by Kvernadze [10, p.185].

From the equiconvergence formula (2.1) and from the identities:

$$S_n(w;f;x) = f(x) - R_n(w;f;x),$$

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = f(x) - R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x),$$

we obtain

$$(2.2) \qquad ||R_n(w;f;x) - R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)||_{C[\Delta(\nu;\frac{\varepsilon}{2})]} = o(1).$$

From an obvious identity [13]

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = S_n(g,\theta),$$

where $x = \cos\theta$, $g(\theta) = f(cos\theta)$ one has

$$R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = R_n(g,\theta).$$

Integrating the last identity with respect to x we obtain

$$(2.3) \qquad [R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} R_n^{(-1)}(g;\theta) + \int R_n^{(-1)}(g;\theta) cos\theta d\theta\,.$$

Multiplying by n the identity (2.3), we get
(2.4)

$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}} \lim_{n\to\infty} nR_n^{(-1)}(g;\theta) + \lim_{n\to\infty} n \int R_n^{(-1)}(g;\theta) cos\theta d\theta\,.$$

Since

$$|n \int R_n^{(-1)}(g;\theta) cos\theta d\theta| \leqslant \int n|R_n^{(-1)}(g;\theta)| d\theta\,,$$

it is enough to estimate the term $n|R_n^{(-1)}(g;\theta)|$.

If $G(\theta) = \dfrac{\pi-\theta}{2}$, $\theta \in (0, 2\pi)$, is the 2π periodic sawtooth function, then the function g can be represented as follows [9]:

$$(2.5) \qquad g_c(\theta) \equiv g(\theta) - \frac{1}{\pi} \sum_{m=0}^{M-1} [g]_m G(\theta - \theta_m),$$

where θ_m and $[g]_m$, $m = 0, 1, ...M-1$, are the locations of discontinuities and the associated jumps of the function g, and g_c is the $2\pi-$ periodic continuous function, which is piecewise smooth on $[-\pi, \pi]$.

Obviously,

$$(2.6) \qquad g_c \in C \cap V_p\,.$$

Continuity of g_c follows from (2.5). Besides, since $G \in V \subset V_p$ and V_p is a linear vector space, $g_c \in V_p$ as well.

It is known that if $g_c \in V_p$, $1 \leqslant p < 2$, then the function g is continous if and only if its Fourier coefficients satisfy the following condition [7]:

$$(2.7) \qquad \sum_{k=n}^{\infty} \left(a_k(f)^2 + b_k(f)^2\right) = o\left(\frac{1}{n}\right)\,.$$

Thus, according to (2.6), (2.7) and Cauchy-Schwartz inequality we have:

$$n|R_n^{(-1)}(g_c;\theta)| \leqslant n\sum_{k=n}^{\infty}\frac{|a_k(g_c)|+|b_k(g_c)|}{k}$$

$$\leqslant \sqrt{2}\,n\left(\sum_{k=n}^{\infty}(a_k(g_c)^2+b_k(g_c)^2\right)^{\frac{1}{2}}\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}$$

$$= n\,o\left(n^{-\frac{1}{2}}\right)\,O\left(n^{-\frac{1}{2}}\right)=o(1), \tag{2.8}$$

uniformly with respect to $\theta \in [-\pi,\pi]$.

As, by means of a change of variables the problem can always be reduced to the case $\theta = 0$, according to [9, p.33] we have

$$nR_n^{(-1)}(G(\theta_m),0) = o(1)\ . \tag{2.9}$$

By use of (2.8) and (2.9) it follows

$$\lim_{n\to\infty} n|R_n^{(-1)}(g;\theta)| = 0\,. \tag{2.10}$$

Using (2.3) and (2.10) we get

$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}}\lim_{n\to\infty} nR_n^{(-1)}(g;\theta)\,. \tag{2.11}$$

Further, using Theorem (F) for $r=0$ we have

$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}}\frac{-1}{\pi}(g(\theta+)-g(\theta-))\,. \tag{2.12}$$

Hence, taking into account that $f(x\pm)=g(\theta\mp)$, $\theta\in[0,\pi]$ in the identity (2.12), we get:

$$\lim_{n\to\infty} n[R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}}\frac{1}{\pi}(f(x+)-f(x-))\,. \tag{2.13}$$

Finally, result follows from the equiconvergence formula (2.2). □

Theorem 2.2. *Let f be a function of bounded p-variation, i.e. $f\in V_p$, $1\leqslant p<2$, which has a finite number of discontinuities such that $fw\in L[-1,1], w\in GJ$. Then the sequence $\{n^2a_n(w;f)\int P_n(w;x)dx\}$ is (C,α), $\alpha > 1-\frac{1}{p}$ summable to $\frac{(1-x^2)^{\frac{1}{2}}}{\pi}(f(x+0)-f(x-0))$ for every $x\in[-1,1]$, where $\{a_n(w;f)\int P_n(w;x)dx\}$ is the n-th term of the integrated Fourier-Jacobi series of f.*

Proof. By $a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$ we denote the n -th term of the Fourier-Tchebycheff series of f [4]. From the equiconvergence formula (2.1) and identities

$$S_n(w;f;x) = S_{n-1}(w;f;x)+a_n(w;f)P_n(w;x),$$

$$S_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x) = S_{n-1}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)+a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

by the triangle inequality we get:

$$(2.14)\qquad ||a_n(w;f)P_n(w;x) - a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)||_{C[\Delta(\nu;\frac{\epsilon}{2})]} = o(1)\,.$$

According to the identity (2.11) we have:

$$\lim_{n\to\infty}(n+1)[R_{n+1}^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} = -(1-x^2)^{\frac{1}{2}}\lim_{n\to\infty}(n+1)R_{n+1}^{(-1)}(g;\theta)\,.$$

Subtracting the last identity from the identity (2.11), we get

$$(2.15)\quad \lim_{n\to\infty} n a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)\int P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)dx = -(1-x^2)^{\frac{1}{2}}\lim_{n\to\infty}(a_n\sin nx - b_n\cos nx) +$$

$$\lim_{n\to\infty}\left([R_n^{(-\frac{1}{2},-\frac{1}{2})}(f;x)]^{(-1)} + (1-x^2)^{\frac{1}{2}}R_n^{(-1)}(g;\theta)\right).$$

The second sumand on the right side of the equation (2.16) tends to zero according to (2.11). Now, multiplaing by n the identity

$$(2.16)\quad \lim_{n\to\infty} n a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)\int P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)dx = -(1-x^2)^{\frac{1}{2}}\lim_{n\to\infty}(a_n\sin nx - b_n\cos nx)\ ,$$

we have:

(2.17)

$$\lim_{n\to\infty} n^2 a_n^{(-\frac{1}{2},-\frac{1}{2})}(f)\int P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)dx = (1-x^2)^{\frac{1}{2}}\lim_{n\to\infty}(nb_n\cos nx - na_n\sin nx)\,.$$

Finaly, the result follows from the Theorem (D) and the equiconvergence formula (2.14)

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ODD-DIMENSIONAL RIEMANNIAN SPACES WITH ALMOST CONTACT AND ALMOST PARACONTACT STRUCTURES

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A B S T R A C T

Riemannian spaces admitting almost contact and almost paracontact structures are studied from the point of view of compositions in spaces with a symmetric a ne connection. Linear connections with torsion preserving by di erentiation the almost (para-)contact structure or the metric tensor are considered.

1. INTRODUCTION

Riemannian spaces with almost contact and almost paracontact structures have been studied by various authors, e.g. [1, 3, 4, 5, 8, 9, 10]. The almost contact structure is an odd-dimensional extension of the complex structure, and the almost structure can be considered as an extension of the almost product structure. By the help of n independent vector elds in [13, 11, 12, 2] an apparatus for studying of spaces endowed with a symmetric a ne connection is constructed. In this work we apply this apparatus to study odd-dimensional Riemannian spaces V2n+1 admitting almost contact and almost paracontact structures. We prove that if these structures are parallel to the Levi-Civita connection of the Riemannian metric the space V2n+1 is a topological product of three di erentiable manifolds Xn Xn X1. We also determine the projecting a nors of the structures and by their help obtain some characteristics of the considered space.

In the last section, we study linear connections with respect to which the structures of the space are parallel. We de ne a connection with torsion which preserves the metric tensor by covariant di erentiation and compute the components of its curvature tensor.

2. Preliminaries

Let V_{2n+1} be a Riemannian space with metric tensor $g_{\alpha\beta}(\overset{\tau}{u})$ and Levi-Civita connection ∇ with Cristoffel symbols $\Gamma^{\sigma}_{\alpha\beta}$. Then, it is known that $\nabla_{\sigma} g_{\alpha\beta} = 0$.

We introduce the following notations

$$\begin{aligned}
&\alpha,\beta,\gamma,\delta,\nu,\sigma,\tau=1,2,...,2n+1,\\
(2.1)\qquad &a,b,c,d,e=1,2,...,2n,\\
&j,k,l,p,q,s=1,2,...,n;\quad \bar{j},\bar{k},\bar{l},\bar{p},\bar{q},\bar{s}=n+1,n+2,...,2n.
\end{aligned}$$

Let $\underset{\alpha}{v}^{\beta}$ $(\alpha=1,2,...,2n+1)$ be independent vector fields satisfying the following conditions:

$$(2.2)\qquad \begin{aligned}
&g_{\alpha\beta}\underset{\sigma}{v}^{\alpha}\underset{\sigma}{v}^{\beta}=1,\quad g_{\alpha\beta}\underset{k}{v}^{\alpha}\underset{\bar{k}}{v}^{\beta}=0,\quad g_{\alpha\beta}\underset{a}{v}^{\alpha}\underset{2n+1}{v}^{\beta}=0,\\
&g_{\alpha\beta}\underset{k}{v}^{\alpha}\underset{s}{v}^{\beta}=\cos\underset{ks}{\omega},\qquad g_{\alpha\beta}\underset{\bar{k}}{v}^{\alpha}\underset{\bar{s}}{v}^{\beta}=\cos\underset{\bar{k}\bar{s}}{\omega}.
\end{aligned}$$

The net defined by the vector fields $\underset{\alpha}{v}^{\beta}$ will be denoted by $\left\{\underset{\alpha}{v}\right\}$. The reciprocal covectors $\overset{\alpha}{v}_{\beta}$ of the vectors $\underset{\alpha}{v}^{\beta}$ are defined by

$$(2.3)\qquad \underset{\sigma}{v}^{\beta}\,\overset{\sigma}{v}_{\alpha}=\delta_{\alpha}^{\beta}\quad\Leftrightarrow\quad \underset{\alpha}{v}^{\sigma}\,\overset{\beta}{v}_{\sigma}=\delta_{\alpha}^{\beta},$$

where δ_{α}^{β} is the identity affinor.

If we choose the net $\left\{\underset{\alpha}{v}\right\}$ to be the coordinate net, we have

$$(2.4)\qquad \begin{aligned}
&\underset{1}{v}^{\beta}\left(\frac{1}{\sqrt{g_{11}}},0,0,...,0\right),\underset{2}{v}^{\beta}\left(0,\frac{1}{\sqrt{g_{22}}},0,...,0\right),...,\underset{2n+1}{v}^{\beta}\left(0,0,...,0,\frac{1}{\sqrt{g_{2n+1\ 2n+1}}}\right);\\
&\overset{1}{v}_{\beta}\left(\sqrt{g_{11}},0,0,...,0\right),\overset{2}{v}_{\beta}\left(0,\sqrt{g_{22}},0,...,0\right),...,\overset{2n+1}{v}_{\beta}\left(0,0,...,0,\sqrt{g_{2n+1\ 2n+1}}\right).
\end{aligned}$$

According to (2.2) and (2.4), in the parameters of the coordinate net $\left\{\underset{\alpha}{v}\right\}$ the matrix of the metric tensor has the form

$$(2.5)\qquad ||g_{\alpha\beta}||=\left\|\begin{matrix} g_{sk} & 0 & 0\\ 0 & g_{\bar{s}\bar{k}} & 0\\ 0 & 0 & g_{2n+1\ 2n+1}\end{matrix}\right\|.$$

From (2.4) and (2.5) it follows that $g_{\alpha\beta}\underset{2n+1}{v}^{\alpha}=\overset{2n+1}{v}_{\beta}$. Also, the following equalities are valid [13]:

$$(2.6)\qquad \nabla_{\sigma}\,\underset{\alpha}{v}^{\beta}=\underset{\alpha}{\overset{\nu}{T}}_{\sigma}\,\underset{\nu}{v}^{\beta},\qquad \nabla_{\sigma}\,\overset{\alpha}{v}_{\beta}=-\underset{\nu}{\overset{\alpha}{T}}_{\sigma}\,\overset{\nu}{v}_{\beta},$$

where $\nabla_{\sigma}\,\underset{\alpha}{v}^{\beta}=\partial_{\sigma}\,\underset{\alpha}{v}^{\beta}+\Gamma_{\sigma\nu}^{\beta}\,\underset{\alpha}{v}^{\nu}$ and $\nabla_{\sigma}\overset{\alpha}{v}_{\beta}=\partial_{\sigma}\,\overset{\alpha}{v}_{\beta}-\Gamma_{\sigma\beta}^{\nu}\,\overset{\alpha}{v}_{\nu}$.

After contracting with $\overset{\tau}{v}_{\beta}$ both sides of the first equality in (2.6) and taking into account (2.3), we obtain

$$(2.7)\qquad \underset{\alpha}{\overset{\tau}{T}}_{\sigma}=\partial_{\sigma}\,\underset{\alpha}{v}^{\beta}\,\overset{\tau}{v}_{\beta}+\Gamma_{\sigma\nu}^{\beta}\,\underset{\alpha}{v}^{\nu}\,\overset{\tau}{v}_{\beta}.$$

According to (2.4), in the parameters of the coordinate net $\left\{\underset{\alpha}{v}\right\}$ equalities (2.7) take the form

$$\text{(2.8)}\qquad \begin{aligned} &\underset{\alpha}{\overset{\tau}{T}}_\sigma = \frac{\sqrt{g_{\tau\tau}}}{\sqrt{g_{\alpha\alpha}}}\Gamma^\tau_{\sigma\alpha} \qquad \text{for } \tau \neq \alpha,\\ &\underset{\alpha}{\overset{\alpha}{T}}_\sigma = \Gamma^\alpha_{\sigma\alpha} - \frac{1}{2}\frac{\partial_\sigma\, g_{\alpha\alpha}}{g_{\alpha\alpha}} \quad \text{(no summing over } \alpha). \end{aligned}$$

Now, let us consider the following affinor [11, 12, 2]:

$$\text{(2.9)}\qquad a^\beta_\alpha = \underset{a}{v}^\beta\, \overset{a}{v}_\alpha - \underset{2n+1}{v}^\beta\, \overset{2n+1}{v}_\alpha .$$

From (2.3) and (2.9) we obtain $a^\beta_\alpha\, a^\sigma_\beta = \delta^\sigma_\alpha$. Hence, the affinor a^β_α defines a composition $X_{2n} \times X_1$ of the basic manifolds X_{2n} and X_1.

The positions (tangent planes) of the basic manifolds X_{2n} and X_1 are denoted by $P(X_{2n})$ and $P(X_1)$, respectively [7].

According to [11, 12], the affinors

$$\overset{1}{a}{}^\beta_\alpha = \frac{1}{2}(\delta^\beta_\alpha + a^\beta_\alpha) = \underset{a}{v}^\beta\, \overset{a}{v}_\alpha, \qquad \overset{2}{a}{}^\beta_\alpha = \frac{1}{2}(\delta^\beta_\alpha - a^\beta_\alpha) = \underset{2n+1}{v}^\beta\, \overset{2n+1}{v}_\alpha$$

are the projecting affinors of the composition $X_{2n} \times X_1$. If v^β is an arbitrary vector, we have $v^\beta = \overset{1}{a}{}^\beta_\alpha\, v^\alpha + \overset{2}{a}{}^\beta_\alpha\, v^\alpha = \overset{1}{V}{}^\beta + \overset{2}{V}{}^\beta$, where $\overset{1}{V}{}^\beta = \overset{1}{a}{}^\beta_\alpha\, v^\alpha \in P(X_{2n})$ and $\overset{2}{V}{}^\beta = \overset{2}{a}{}^\beta_\alpha\, v^\alpha \in P(X_1)$. Obviously, $\underset{a}{v}^\alpha \in P(X_{2n})$, and $\underset{2n+1}{v}^\alpha \in P(X_1)$.

Let $X_a \times X_b$ $(a+b=n)$ be an arbitrary composition in the Riemannian space V_n, and $P(X_a)$ and $P(X_b)$ be the positions of the differentiable manifolds X_a and X_b, respectively. According to [7], the composition $X_a \times X_b$ is of the type (c,c), i.e. (Cartesian, Cartesian), if the positions $P(X_a)$ and $P(X_b)$ are translated parallelly along any line in the space V_n.

3. Almost contact and almost paracontact structures on V_{2n+1}

Let us consider the following affinors

$$\text{(3.1)}\qquad \underset{\lambda}{b}{}^\beta_\alpha = \lambda\left(\underset{k}{v}^\beta\, \overset{k}{v}_\alpha - \underset{\bar{k}}{v}^\beta\, \overset{\bar{k}}{v}_\alpha\right),$$

where $\lambda = 1, i$ (i is the imaginary unit, i.e. $i^2=-1$). According to (2.3) and (3.1) we have $\underset{\lambda}{b}{}^\beta_\alpha\, \underset{2n+1}{v}^\alpha = 0$ and $\underset{\lambda}{b}{}^\beta_\alpha\, \overset{2n+1}{v}_\beta = 0$.

Let $\lambda = 1$. From (2.3) and (3.1) we obtain

$$\underset{1}{b}{}^\beta_\alpha\, \underset{1}{b}{}^\sigma_\beta = \delta^\sigma_\alpha - \underset{2n+1}{v}^\sigma\, \overset{2n+1}{v}_\alpha,$$

i.e. the affinor $\underset{1}{b}{}^\beta_\alpha$ defines an almost paracontact structure on V_{2n+1}.

In the parameters of the coordinate net, it is easy to prove that

$$\text{(3.2)}\qquad g_{\sigma\nu}\, \underset{1}{b}{}^\sigma_\alpha\, \underset{1}{b}{}^\nu_\beta = g_{\alpha\beta} - \overset{2n+1}{v}_\alpha\, \overset{2n+1}{v}_\beta,$$

i.e. the almost paracontact structure $\underset{1}{b}_\alpha^\beta$ is compatible with the Riemannian metric $g_{\alpha\beta}$, and hence V_{2n+1} is an almost paracontact Riemannian manifold [1, 8].

In the case $\lambda = i$ the affinor (3.1) defines an almost contact structure in V_{2n+1} which is not compatible with the Riemannian metric $g_{\alpha\beta}$, i.e. (3.2) does not hold for $\underset{i}{b}_\alpha^\beta$.

Theorem 3.1. *The affinor $\underset{\lambda}{b}_\alpha^\beta$ is parallel to the Levi-Civita connection ∇, i.e. $\nabla_\sigma \underset{\lambda}{b}_\alpha^\beta = 0$, iff the coefficients of the derivative equations (2.6) satisfy*

$$\underset{k}{\overset{\bar{s}}{T}}_\sigma = \underset{\bar{k}}{\overset{s}{T}}_\sigma = 0, \qquad \underset{2n+1}{\overset{a}{T}}{}_\sigma = \underset{a}{\overset{2n+1}{T}}{}_\sigma = 0. \tag{3.3}$$

Proof. Let

$$\nabla_\sigma \underset{\lambda}{b}_\alpha^\beta = 0. \tag{3.4}$$

According to (2.6) and (3.1), equality (3.4) takes the form

$$\underset{k}{\overset{\nu}{T}}_\sigma \underset{\nu}{v}^\beta \overset{k}{v}_\alpha - \underset{\nu}{\overset{k}{T}}_\sigma \underset{k}{v}^\beta \overset{\nu}{v}_\alpha - \underset{\bar{k}}{\overset{\nu}{T}}_\sigma \underset{\nu}{v}^\beta \overset{\bar{k}}{v}_\alpha + \underset{\nu}{\overset{\bar{k}}{T}}_\sigma \underset{\bar{k}}{v}^\beta \overset{\nu}{v}_\alpha = 0. \tag{3.5}$$

After contracting (3.5) with $\underset{s}{\nu}^\alpha$, $\underset{\bar{s}}{v}^\alpha$ and $\underset{2n+1}{v}{}^\alpha$, we obtain the following equalities which are equivalent to (3.5):

$$\begin{gathered} 2\,\underset{s}{\overset{\bar{k}}{T}}_\sigma \underset{\bar{k}}{v}^\beta + \underset{s}{\overset{2n+1}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta = 0, \qquad 2\,\underset{\bar{s}}{\overset{k}{T}}_\sigma \underset{k}{v}^\beta + \underset{\bar{s}}{\overset{2n+1}{T}}{}_\sigma \underset{2n+1}{v}{}^\beta = 0, \\ \underset{2n+1}{\overset{k}{T}}{}_\sigma \underset{k}{v}^\beta - \underset{2n+1}{\overset{\bar{k}}{T}}{}_\sigma \underset{\bar{k}}{v}^\beta = 0. \end{gathered} \tag{3.6}$$

From the independency of the vectors $\underset{\nu}{v}^\beta$ it follows that equalities (3.6) are equivalent to conditions (3.3) which proves the statement. □

Let us note that manifolds satisfying (3.4) are contact and paracontact analogues to Kähler manifolds.

Corollary 3.1. *If $\nabla_\sigma \underset{\lambda}{b}_\alpha^\beta = 0$, in the parameters of the net $\left\{\underset{\alpha}{v}\right\}$, the Christoffel symbols $\Gamma_{\alpha\beta}^\nu$ satisfy*

$$\Gamma_{\sigma s}^{\bar{k}} = 0, \quad \Gamma_{\sigma\bar{s}}^{k} = 0, \quad \Gamma_{\sigma 2n+1}^{a} = 0, \quad \Gamma_{\sigma a}^{2n+1} = 0. \tag{3.7}$$

Proof. According to (2.8), equalities (3.3) take the form (3.7). □

Corollary 3.2. *If $\nabla_\sigma \underset{\lambda}{b}_\alpha^\beta = 0$, the composition $X_{2n} \times X_1$ defined by the affinor (2.9), is of the type (c, c).*

Proof. Having in mind (3.4), equalities (3.7) hold.
Then, according to [7], from $\Gamma_{\sigma 2n+1}^{a} = \Gamma_{\sigma a}^{2n+1} = 0$ it follows that the composition $X_{2n} \times X_1$ is of the type (c, c). □

From (2.5) it follows that the composition $X_{2n} \times X_1$ is orthogonal. The coordinate net $\left\{\underset{\alpha}{v}\right\}$ gives rise to coordinates which are adapted to the composition $X_{2n} \times X_1$. In accordance to [6], the line element of the space V_{2n+1} is of the form

$$\mathrm{d}s^2 = g_{ab}(\overset{c}{u})\mathrm{d}\overset{a}{u}\mathrm{d}\overset{b}{u} + g_{2n+1\ 2n+1}(\overset{2n+1}{u})\mathrm{d}(\overset{2n+1}{u})^2, \tag{3.8}$$

where g_{ab} is the metric tensor of the manifold X_{2n}.

Theorem 3.2. *If condition (3.4) holds, the Riemannian space X_{2n} is a space of the composition $X_n \times \overline{X}_n$ with line element defined in the parameters of the net $\left\{\underset{\alpha}{v}\right\}$ by*

$$\mathrm{d}s^2 = g_{ks}(\overset{i}{u})\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} + g_{\bar{k}\bar{s}}(\overset{\bar{i}}{u})\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u}. \tag{3.9}$$

Proof. The tensors $\underset{\lambda}{b}{}_a^d$, $\nabla_c \underset{\lambda}{b}{}_a^d$ and g_{ab} are the full projections of the tensors $\underset{\lambda}{b}{}_\alpha^\beta$, $\nabla_\sigma \underset{\lambda}{b}{}_\alpha^\beta$ and $g_{\alpha\beta}$, respectively, over the positions $P(X_{2n})$.

From (3.1) it follows that $\underset{\lambda}{b}{}_a^d \underset{\lambda}{b}{}_d^c = \pm\delta_a^c$. Hence, the affinor $\underset{\lambda}{b}{}_a^d$ defines a composition $X_n \times \overline{X}_n$ in the manifold X_{2n}. Because of the condition $\nabla_c \underset{\lambda}{b}{}_a^d = 0$, the composition $X_n \times \overline{X}_n$ is of the type (c,c) [7]. From (2.5) it follows that the composition $X_n \times \overline{X}_n$ is orthogonal. Then, according to [6], the line element of $X_n \times \overline{X}_n$ is of the form (3.9). □

Let $P(X_n)$ and $P(\overline{X}_n)$ are the positions of the differentiable manifolds X_n and $\overline{X}_n$, respectively. The projecting affinors of the composition $X_n \times \overline{X}_n$ are:

$$\overset{1}{b}{}_\alpha^\beta = \lambda \underset{k}{v}^\beta \overset{k}{v}_\alpha, \qquad \overset{2}{b}{}_\alpha^\beta = \lambda \underset{\bar{k}}{v}^\beta \overset{\bar{k}}{v}_\alpha.$$

For an arbitrary vector $w^\alpha \in P(X_{2n})$ we have $w^\beta = \overset{1}{b}{}_\alpha^\beta w^\alpha + \overset{2}{b}{}_\alpha^\beta w^\alpha = \overset{1}{W}{}^\beta + \overset{2}{W}{}^\beta$, where $\overset{1}{W}{}^\beta = \overset{1}{b}{}_\alpha^\beta w^\alpha \in P(X_n)$, and $\overset{2}{W}{}^\beta = \overset{2}{b}{}_\alpha^\beta w^\alpha \in P(\overline{X}_n)$. Obviously, $\underset{k}{v}^\beta \in P(X_n)$, and $\underset{\bar{k}}{v}^\beta \in P(\overline{X}_n)$.

The following statements are immediate consequences of our results:

Proposition 3.1. *If condition (3.4) holds, the Riemannian space V_{2n+1} is a topological product of three basic differentiable manifolds X_n, $\overline{X}_n$ and X_1, i.e. V_{2n+1} is a space of the composition $X_n \times \overline{X}_n \times X_1$.* ■

Proposition 3.2. *If (3.4) holds, in the parameters of the coordinate net $\left\{\underset{\alpha}{v}\right\}$ the line element of the space V_{2n+1} is of the form*

$$\mathrm{d}s^2 = g_{ks}(\overset{i}{u})\mathrm{d}\overset{k}{u}\mathrm{d}\overset{s}{u} + g_{\bar{k}\bar{s}}(\overset{\bar{i}}{u})\mathrm{d}\overset{\bar{k}}{u}\mathrm{d}\overset{\bar{s}}{u} + g_{2n+1\ 2n+1}(\overset{2n+1}{u})\mathrm{d}(\overset{2n+1}{u})^2. \tag{3.10}$$ ■

Now we will prove the following theorem.

Theorem 3.3. *Condition (3.4) is equivalent to the following:*

$$\overset{1}{b}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{1}{b}{}^{\beta}_{\sigma}=0,\qquad \overset{2}{b}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{2}{b}{}^{\beta}_{\sigma}=0,\qquad \overset{2}{a}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{2}{a}{}^{\beta}_{\sigma}=0, \tag{3.11}$$

where $\overset{1}{b}{}^{\sigma}_{\nu}$, $\overset{2}{b}{}^{\sigma}_{\nu}$ *and* $\overset{2}{a}{}^{\sigma}_{\nu}$ *are the projecting affinors of the composition* $X_n\times\overline{X}_n\times X_1$.

Proof. Because of $\overset{1}{b}{}^{\sigma}_{\nu}=\lambda\ \underset{k}{v}{}^{\sigma}\ \overset{k}{v}_{\nu}$, $\overset{2}{b}{}^{\sigma}_{\nu}=\lambda\ \underset{\bar k}{v}{}^{\sigma}\ \overset{\bar k}{v}_{\nu}$ and $\overset{2}{a}{}^{\sigma}_{\nu}=\underset{2n+1}{v}{}^{\sigma}\ \overset{2n+1}{v}{}_{\nu}$, we obtain

$$\begin{aligned}
&\overset{1}{b}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{1}{b}{}^{\beta}_{\sigma}=\pm\ \underset{k}{v}{}^{\sigma}\ \overset{k}{v}_{\nu}\ \nabla_{\alpha}\left(\underset{s}{v}{}^{\beta}\ \overset{s}{v}_{\sigma}\right),\\
&\overset{2}{b}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{2}{b}{}^{\beta}_{\sigma}=\pm\ \underset{\bar k}{v}{}^{\sigma}\ \overset{\bar k}{v}_{\nu}\ \nabla_{\alpha}\left(\underset{\bar s}{v}{}^{\beta}\ \overset{\bar s}{v}_{\sigma}\right),\\
&\overset{2}{a}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{2}{a}{}^{\beta}_{\sigma}=\underset{2n+1}{v}{}^{\sigma}\ \overset{2n+1}{v}{}_{\nu}\ \nabla_{\alpha}\left(\underset{2n+1}{v}{}^{\beta}\ \overset{2n+1}{v}{}_{\sigma}\right).
\end{aligned} \tag{3.12}$$

According to (2.6) and (3.12), we get

$$\begin{aligned}
&\overset{1}{b}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{1}{b}{}^{\beta}_{\sigma}=\pm\left(\underset{k}{\overset{\bar s}{T}}{}_{\alpha}\ \underset{\bar s}{v}{}^{\beta}+\underset{k}{\overset{2n+1}{T}}{}_{\alpha}\ \underset{2n+1}{v}{}^{\beta}\right)\ \overset{k}{v}_{\nu},\\
&\overset{2}{b}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{2}{b}{}^{\beta}_{\sigma}=\pm\left(\underset{\bar k}{\overset{s}{T}}{}_{\alpha}\ \underset{s}{v}{}^{\beta}+\underset{\bar k}{\overset{2n+1}{T}}{}_{\alpha}\ \underset{2n+1}{v}{}^{\beta}\right)\ \overset{\bar k}{v}_{\nu},\\
&\overset{2}{a}{}^{\sigma}_{\nu}\ \nabla_{\alpha}\ \overset{2}{a}{}^{\beta}_{\sigma}=\underset{2n+1}{\overset{a}{T}}{}_{\alpha}\ \underset{a}{v}{}^{\beta}\ \overset{2n+1}{v}{}_{\nu}.
\end{aligned} \tag{3.13}$$

From (3.13) it follows that conditions (3.11) hold iff conditions (3.3) hold, too. And, according to Theorem 3.1, (3.3) are equivalent to condition (3.4). Then, (3.4) and (3.11) are also equivalent which completes the proof. □

In accordance to (3.7), for the components of the curvature tensor $R_{\alpha\beta\sigma}{}^{\nu}=\partial_{\alpha}\Gamma^{\nu}_{\beta\sigma}-\partial_{\beta}\Gamma^{\nu}_{\alpha\sigma}+\Gamma^{\nu}_{\alpha\delta}\Gamma^{\delta}_{\beta\sigma}-\Gamma^{\nu}_{\beta\delta}\Gamma^{\delta}_{\alpha\sigma}$ we obtain

$$R_{\alpha ks}{}^{\bar j}=R_{ks\alpha}{}^{\bar j}=R_{\alpha\bar k\bar s}{}^{j}=R_{\bar k\bar s\alpha}{}^{j}=R_{\alpha ab}{}^{2n+1}=R_{ab\alpha}{}^{2n+1}=0. \tag{3.14}$$

4. Transformations of linear connections

4.1. Linear connections with torsion. Let us consider the linear connection

$${}^{1}\Gamma^{\nu}_{\alpha\beta}=\Gamma^{\nu}_{\alpha\beta}+S^{\nu}_{\alpha\beta}, \tag{4.1}$$

where $S^{\nu}_{\alpha\beta}$ is the deformation tensor. The covariant derivative and the curvature tensor with respect to ${}^{1}\Gamma$ are denoted by ${}^{1}\nabla$ and ${}^{1}R$.

Theorem 4.1. *The affinors (3.1) are parallel to* ∇ *and* $^1\nabla$ *iff in parameters of the net* $\left\{ \underset{\alpha}{v} \right\}$ *the tensor* $S^{\nu}_{\alpha\beta}$ *satisfies*

$$S^{s}_{\alpha\bar{k}} = S^{s}_{\alpha 2n+1} = S^{\bar{s}}_{\alpha k} = S^{\bar{s}}_{\alpha 2n+1} = S^{2n+1}_{\alpha\alpha} = 0. \tag{4.2}$$

Proof. Let conditions (3.4) hold and let

$$^1\nabla_{\sigma}\ \underset{\lambda}{b}^{\beta}_{\alpha} = 0. \tag{4.3}$$

According to (4.1), we have $^1\nabla_{\sigma}\ \underset{\lambda}{b}^{\beta}_{\alpha} = \nabla_{\sigma}\ \underset{\lambda}{b}^{\beta}_{\alpha} + S^{\beta}_{\sigma\nu}\ \underset{\lambda}{b}^{\nu}_{\alpha} - S^{\nu}_{\sigma\alpha}\ \underset{\lambda}{b}^{\beta}_{\nu}$, from where it follows that equalities (3.4) and (4.3) hold iff

$$P^{\beta}_{\sigma\alpha} = S^{\beta}_{\sigma\nu}\ \underset{\lambda}{b}^{\nu}_{\alpha} - S^{\nu}_{\sigma\alpha}\ \underset{\lambda}{b}^{\beta}_{\nu} = 0. \tag{4.4}$$

We choose $\left\{ \underset{\alpha}{v} \right\}$ for the coordinate net. In its parameters of the net, the matrix of the affinor $\underset{\lambda}{b}^{\beta}_{\alpha}$ has the form

$$\left\| \underset{\lambda}{b}^{\beta}_{\alpha} \right\| = \left\| \begin{matrix} \lambda\delta^{k}_{s} & 0 & 0 \\ 0 & -\lambda\delta^{\bar{k}}_{\bar{s}} & \vdots \\ 0 & \dots & 0 \end{matrix} \right\|. \tag{4.5}$$

From (4.4) and (4.5) we compute the following non-zero components of P:

$$\begin{array}{lll} P^{j}_{s\bar{k}} = -2\lambda S^{j}_{s\bar{k}}, & P^{j}_{\bar{s}\bar{k}} = -2\lambda S^{j}_{\bar{s}\bar{k}}, & P^{j}_{s2n+1} = -\lambda S^{j}_{s2n+1}, \\ P^{j}_{2n+1,2n+1} = -\lambda S^{j}_{2n+1,2n+1}, & P^{j}_{\bar{s}2n+1} = -\lambda S^{j}_{\bar{s}2n+1}, & P^{j}_{2n+1\bar{s}} = -2\lambda S^{j}_{2n+1\bar{s}}, \\ P^{\bar{j}}_{\bar{s}k} = 2\lambda S^{\bar{j}}_{\bar{s}k}, & P^{\bar{j}}_{sk} = 2\lambda S^{\bar{j}}_{sk}, & P^{\bar{j}}_{\bar{s}2n+1} = \lambda S^{\bar{j}}_{\bar{s}2n+1} \\ P^{\bar{j}}_{2n+1,2n+1} = \lambda S^{\bar{j}}_{2n+1,2n+1}, & P^{\bar{j}}_{s2n+1} = \lambda S^{\bar{j}}_{s2n+1}, & P^{\bar{j}}_{2n+1s} = 2\lambda S^{\bar{j}}_{2n+1s}, \\ P^{2n+1}_{sk} = \lambda S^{2n+1}_{sk}, & P^{2n+1}_{\bar{s}k} = \lambda S^{2n+1}_{\bar{s}k}, & P^{2n+1}_{s\bar{k}} = -\lambda S^{2n+1}_{k\bar{s}}, \\ P^{2n+1}_{\bar{s}\bar{k}} = -\lambda S^{2n+1}_{\bar{s}\bar{k}}, & P^{2n+1}_{2n+1s} = \lambda S^{2n+1}_{2n+1s}, & P^{2n+1}_{2n+1\bar{s}} = -\lambda S^{2n+1}_{2n+1\bar{s}}, \end{array} \tag{4.6}$$

Then, according to (4.6), equalities (4.4) hold iff (4.2) hold, too. □

From (4.1) and (4.2) we get the non-zero components of $^1\Gamma$ expressed by the components of Γ and S:

$\| \ \lambda\delta^{\kappa}_{-} \quad 0 \quad 0 \ \|$

$$\begin{array}{lll} ^1\Gamma^{j}_{sk} = \Gamma^{j}_{sk} + S^{j}_{sk}, & ^1\Gamma^{j}_{\bar{k}s} = S^{j}_{\bar{k}s}, & ^1\Gamma^{j}_{2n+1s} = S^{j}_{2n+1s} \\ ^1\Gamma^{\bar{j}}_{\bar{s}\bar{k}} = \Gamma^{\bar{j}}_{\bar{s}\bar{k}} + S^{\bar{j}}_{\bar{s}\bar{k}}, & ^1\Gamma^{\bar{j}}_{k\bar{s}} = S^{\bar{j}}_{k\bar{s}}, & ^1\Gamma^{\bar{j}}_{2n+1\bar{s}} = S^{\bar{j}}_{2n+1\bar{s}}, \\ ^1\Gamma^{2n+1}_{s2n+1} = S^{2n+1}_{s2n+1}, & ^1\Gamma^{2n+1}_{\bar{s}2n+1} = S^{2n+1}_{\bar{s}2n+1}, & ^1\Gamma^{2n+1}_{2n+1,2n+1} = S^{2n+1}_{2n+1,2n+1}. \end{array} \tag{4.7}$$

Having in mind (4.7), we compute the following components of the curvature tensor $^1R_{\alpha\beta\sigma}{}^{\nu}$:

$$ {}^1R_{\alpha s k}{}^{\bar{j}} = {}^1R_{\alpha \bar{s} \bar{k}}{}^{j} = {}^1R_{\alpha a b}^{2n+1} = 0, $$

$$ {}^1R_{k s \alpha}{}^{\bar{j}} = 2\left(\partial_{[k} S^{\bar{j}}_{s]\alpha} + S^{\bar{j}}_{[k|\bar{l}|} S^{\bar{l}}_{s]\alpha}\right), \quad {}^1R_{\bar{k}\bar{s}\alpha}{}^{j} = 2\left(\partial_{[\bar{k}} S^{j}_{\bar{s}]\alpha} + S^{j}_{[\bar{k}|l|} S^{l}_{\bar{s}]\alpha}\right), $$

$$ {}^1R_{ab\alpha}^{2n+1} = 2\left(\partial_{[a} S^{2n+1}_{b]\alpha} + S^{2n+1}_{[a|2n+1|} S^{2n+1}_{b]\alpha}\right), $$

$$ {}^1R_{skl}{}^{j} = R_{skl}{}^{j} + 2\left(\partial_{[s} S^{j}_{k]l} + \Gamma^{j}_{[s|p|} S^{p}_{k]l} + S^{j}_{[s|p|}\Gamma^{p}_{k]l} + S^{j}_{[s|p|} S^{p}_{k]l}\right), $$

$$ {}^1R_{\bar{s}\bar{k}\bar{l}}{}^{\bar{j}} = R_{\bar{s}\bar{k}\bar{l}}{}^{\bar{j}} + 2\left(\partial_{[\bar{s}} S^{\bar{j}}_{\bar{k}]\bar{l}} + \Gamma^{\bar{j}}_{[\bar{s}|\bar{p}|} S^{\bar{p}}_{\bar{k}]\bar{l}} + S^{\bar{j}}_{[\bar{s}|\bar{p}|}\Gamma^{\bar{p}}_{\bar{k}]\bar{l}} + S^{\bar{j}}_{[\bar{s}|\bar{p}|} S^{\bar{p}}_{\bar{k}]\bar{l}}\right). $$

4.2. A metric connection. Let V_{2n+1} be a space with $\nabla_\sigma \underset{\lambda}{b}{}_\alpha^\beta = 0$, and let us consider the connection

(4.8) $$ {}^2\Gamma^{\nu}_{\alpha\beta} = \Gamma^{\nu}_{\alpha\beta} + \bar{S}^{\nu}_{\alpha\beta}, $$

where

(4.9) $$ \bar{S}^{\nu}_{\alpha\beta} = \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\alpha \, g_{\beta\delta} \sum_{k=1}^{n} \left(\underset{k}{v}{}^\delta \underset{k+n}{v}{}^\nu - \underset{k}{v}{}^\nu \underset{k+n}{v}{}^\delta \right). $$

The covariant derivative and the curvature tensor with respect to the connection ${}^2\Gamma$ are denoted by ${}^2\nabla$ and ${}^2R$.

Theorem 4.2. *The metric tensor of the space V_{2n+1} is parallel to the connection ${}^2\Gamma$, i.e.*

(4.10) $$ {}^2\nabla_\sigma g_{\alpha\beta} = 0\,. $$

Proof. From (4.8) and (4.10) we get

(4.11) $$ {}^2\nabla_\sigma g_{\alpha\beta} = \nabla_\sigma g_{\alpha\beta} - \bar{S}^{\nu}_{\sigma\alpha}\, g_{\nu\beta} - \bar{S}^{\nu}_{\sigma\beta}\, g_{\nu\alpha}. $$

Let us consider the tensor

(4.12) $$ T_{\sigma\alpha\beta} = \bar{S}^{\nu}_{\sigma\alpha} g_{\nu\beta}. $$

According to (4.9) and (4.12), we have

(4.13) $$ T_{\sigma\alpha\beta} = \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma \, g_{\alpha\delta} \sum_{k=1}^{n} \left(\underset{k}{v}{}^\nu \underset{k+n}{v}{}^\delta - \underset{k}{v}{}^\delta \underset{k+n}{v}{}^\nu \right) g_{\nu\beta}. $$

In the parameters of the coordinate net $\left\{ \underset{\alpha}{v} \right\}$ we obtain

$$ T_{\sigma\alpha\beta} = \sum_{\tau=1}^{2n+1} \overset{\tau}{v}_\sigma \sum_{k=1}^{n} \frac{1}{\sqrt{g_{kk}}\sqrt{g_{k+n\ k+n}}} \left(g_{\alpha\ k+n}\ g_{\beta k} - g_{\beta\ k+n}\ g_{\alpha k} \right), $$

from where it follows that

(4.14) $$T_{\sigma(\alpha\beta)} = 0.$$

Then, (4.11), (4.12) and (4.14) imply (4.10). □

By (2.4) and (4.9) we obtain the components of the deformation tensor $\bar{S}$ of ${}^2\nabla$ and then by (3.7) and (4.8) we get the non-zero Christoffel symbols of ${}^2\nabla$ in the parameters of the coordinate net as follows:

(4.15)
$$\begin{aligned}
&{}^2\Gamma^{j}_{k\ n+s} = -\frac{\sqrt{g_{kk}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\ n+j}}}\ g_{n+s\ n+j},\\
&{}^2\Gamma^{j}_{n+k\ n+s} = -\frac{\sqrt{g_{n+k\ n+k}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\ n+j}}}\ g_{n+s\ n+j},\\
&{}^2\Gamma^{n+j}_{sk} = \frac{\sqrt{g_{ss}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\ n+j}}}\ g_{jk},\\
&{}^2\Gamma^{n+j}_{n+s\ k} = \frac{\sqrt{g_{n+s\ n+s}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\ n+j}}}\ g_{jk},\\
&{}^2\Gamma^{j}_{2n+1\ n+s} = -\frac{\sqrt{g_{2n+1\ 2n+1}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\ n+j}}}\ g_{n+s\ n+j},\\
&{}^2\Gamma^{n+j}_{2n+1\ k} = \frac{\sqrt{g_{2n+1\ 2n+1}}}{\sqrt{g_{jj}}\sqrt{g_{n+j\ n+j}}}\ g_{jk}.
\end{aligned}$$

By (4.15) we compute the components of the curvature tensor ${}^2R$, for example

(4.16)
$$\begin{aligned}
&{}^2R_{skp}{}^{j} = R_{skp}{}^{j}, \qquad {}^2R_{\bar{s}\bar{k}\bar{p}}{}^{\bar{j}} = R_{\bar{s}\bar{k}\bar{p}}{}^{\bar{j}}, \qquad {}^2R^{2n+1}_{abc} = 0,\\
&{}^2R_{pk\ n+s}{}^{j} = \frac{g_{n+s\ n+s}}{\sqrt{g_{n+j\ n+j}}}\left(\partial_k\frac{\sqrt{g_{pp}}}{\sqrt{g_{jj}}} - \partial_p\frac{\sqrt{g_{kk}}}{\sqrt{g_{jj}}}\right) + \sqrt{g_{pp}}\sum_{l=1}^{n}\Gamma^{j}_{kl}\frac{g_{n+s\ n+l}}{\sqrt{g_{ll}}\sqrt{g_{n+l\ n+l}}}\\
&\qquad - \sqrt{g_{kk}}\sum_{l=1}^{n}\Gamma^{j}_{pl}\frac{g_{n+s\ n+l}}{\sqrt{g_{ll}}\sqrt{g_{n+l\ n+l}}},\\
&{}^2R^{n+j}_{2n+1\ ks} = \frac{\sqrt{g_{2n+1\ 2n+1}}}{\sqrt{g_{n+j\ n+j}}}\left(\frac{1}{\sqrt{g_{jj}}}g_{lj}\Gamma^{l}_{ks} - \partial_k\frac{g_{sj}}{\sqrt{g_{jj}}}\right).
\end{aligned}$$

As an example we consider a 5-dimensional Riemannian space V_5. The matrix (2.5) has the form

(4.17)
$$||g_{\alpha\beta}|| = \left\| \begin{matrix} g_{sk} & 0 & 0\\ 0 & g_{\bar{s}\bar{k}} & 0\\ 0 & 0 & g_{55} \end{matrix} \right\|,$$

where $j,k,s = 1,2$, $\bar{j},\bar{k},\bar{s} = 3,4$.

In the parameters of the net $\left\{\underset{\alpha}{v}\right\}$ the line element if given by

(4.18)
$$ds^2 = g_{sk}(\overset{j}{u})d\overset{k}{u}d\overset{s}{u} + g_{\bar{k}\bar{s}}(\overset{\bar{j}}{u})d\overset{\bar{k}}{u}d\overset{\bar{s}}{u} + g_{55}(\overset{5}{u})d\overset{5}{u}{}^2.$$

From the last one of the equalities (4.16) we get

(4.19)
$${}^2R_{512}{}^{3} = \frac{\sqrt{g_{55}}}{\sqrt{g_{33}}}\left(\frac{1}{\sqrt{g_{11}}}g_{l1}\Gamma^{l}_{12} - \partial_1\frac{g_{12}}{\sqrt{g_{11}}}\right).$$

Since $g_{l1}\Gamma^{l}_{12} = \frac{1}{2}\partial_2 g_{11}$, (4.19) implies

(4.20)
$${}^2R_{512}{}^{3} = \frac{\sqrt{g_{55}}}{\sqrt{g_{33}}}\left(\frac{1}{2\sqrt{g_{11}}}\partial_2 g_{11} - \partial_1\frac{g_{12}}{\sqrt{g_{11}}}\right).$$

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MIKUSI‹SKI'S OPERATIONAL CALCULUS APPROACH TO THE DISTRIBUTIONAL STIELTJES TRANSFORM

DENNIS NEMZER

A B S T R A C T

We consider a space M which was introduced by Yosida to provide a simpli ed version for Mikusi«ski operational calculus. The classical Stieltjes is extended to a subspace of M and then studied. Some Abelian type theorems are presented.

1. INTRODUCTION

The ring of continuous complex-valued functions on the real line which vanish on $(-\infty, 0)$, denoted by $C_+(\mathbb{R})$, with addition and convolution has no zero divisors by Titchmarch's theorem. The quotient field of $C_+(\mathbb{R})$ is called the field of Mikusiński operators [6].

Yosida [10] constructed a space $\mathcal{M}$ in order to provide a simplified version for Mikusiński's operational calculus without using Titchmarch's convolution theorem. Even though the space $\mathcal{M}$ does not give the full space of Mikusiński operators, it contains many of the important operators needed for applications.

In this note, we use the space $\mathcal{M}(r) \subset \mathcal{M}$ to extend the classical Stieltjes transform. It turns out that $\mathcal{M}(r)$ is isomorphic to the space of distributions $J'(r)$. Roughly speaking, a distribution T, which is supported on $[0, \infty)$, is in $J'(r)$ provided there exist $k \in \mathbb{N}$ and a locally integrable function f satisfying a growth condition at infinity such that T is the k^{th} distributional derivative of f.

The space $J'(r)$, and variations of $J'(r)$, have been investigated by several authors [2, 4, 5, 7, 8, 9] in regards to extending the Stieltjes transform.

While the construction of $J'(r)$ requires a space of testing functions, the concept of a dual space, and functional analysis, the construction of $\mathcal{M}(r)$ is algebraic, elementary, and only requires elementary calculus.

Let $C_+(\mathbb{R})$ denote the space of all continuous functions on $\mathbb{R}$ which vanish on the interval $(-\infty, 0)$.

For f, $g \in C_+(\mathbb{R})$, the convolution is given by

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx. \tag{2.1}$$

Let H denote the Heaviside function. That is, $H(t) = 1$ for $t \geq 0$ and zero otherwise. For each $n \in \mathbb{N}$, we denote by H^n the function $H * \cdots * H$ where H is repeated n times. The space $\mathcal{M}$ is defined as follows.

$$\mathcal{M} = \left\{ \frac{f}{H^k} : f \in C_+(\mathbb{R}), k \in \mathbb{N} \right\}.$$

Two elements of $\mathcal{M}$ are equal, denoted $\frac{f}{H^n} = \frac{g}{H^m}$, if and only if $H^m * f = H^n * g$.

Addition, multiplication, and scalar multiplication are defined in the natural way, and $\mathcal{M}$ with these operations is a commutative algebra with identity $\delta = \frac{H^2}{H^2}$.

$$\frac{f}{H^n} + \frac{g}{H^m} = \frac{H^m * f + H^n * g}{H^{n+m}} \tag{2.2}$$

$$\frac{f}{H^n} * \frac{g}{H^m} = \frac{f * g}{H^{n+m}} \tag{2.3}$$

$$\alpha \frac{f}{H^n} = \frac{\alpha f}{H^n}, \quad \alpha \in \mathbb{C}. \tag{2.4}$$

The generalized derivative is defined as follows.
Let $W = \frac{f}{H^k} \in \mathcal{M}$. Then, $DW = \frac{f}{H^{k+1}}$.

Remark 2.1. *For the construction of $\mathcal{M}$, the space of locally integrable functions which vanish on $(-\infty, 0)$ could have been used instead of $C_+(\mathbb{R})$. Also notice by identifying $f \in L^1_{loc}(\mathbb{R}^+)$ with $\frac{H*f}{H} \in \mathcal{M}$, $L^1_{loc}(\mathbb{R}^+)$ can be considered a subspace of $\mathcal{M}$.*

3. Stieltjes Transform

For $k = 0, 1, 2, \ldots$

$$\mathcal{M}_k(r) = \left\{ \frac{f}{H^k} \in \mathcal{M} : f(t)\, t^{-r-k+\alpha} \text{ is bounded as } t \to \infty \text{ for some } \alpha > 0 \right\} \tag{3.1}$$

$$\mathcal{M}(r) = \bigcup_{k=0}^{\infty} \mathcal{M}_k(r) \tag{3.2}$$

Let $W \in \mathcal{M}(r)$. That is, $W = \frac{f}{H^k} \in \mathcal{M}_k(r)$, for some $k \in \mathbb{N}$. For $r > -1$, define the Stieltjes transform of index r by

$$\Lambda_z^r W = (r+1)_k \int_0^\infty \frac{f(t)}{(t+z)^{r+k+1}}\, dt, \quad z \in \mathbb{C}\backslash(-\infty, 0], \tag{3.3}$$

where $(r+1)_k = \frac{\Gamma(r+k+1)}{\Gamma(r+1)} = (r+1)(r+2)\cdots(r+k)$ and Γ is the gamma function.

Remark 3.1.

(1) *The definition for the Stieltjes transform is well-defined. This follows by observing the following. First,* $\frac{f}{H^k} = \frac{g}{H^n}$ $(n \geq k)$ *if and only if* $g = H^{n-k} * f$. *Also, for* $m \in \mathbb{N}$,

$$\Lambda_z^r\left(\frac{f}{H^k}\right) = \Lambda_z^r\left(\frac{H^m * f}{H^{m+k}}\right),\ z \in \mathbb{C}\backslash(-\infty, 0].$$

(2) *Notice that the Stieltjes transform* Λ_z^r *is consistent with the classical Stieltjes transform* S_z^r. *That is, if* $f \in L^1_{loc}(\mathbb{R}^+)$ *such that* f *satisfies the growth condition in (3.1) with* $k = 0$, *then* $S_z^r f = \Lambda_z^r\left(\frac{H*f}{H}\right)$, *where* $S_z^r f = \int_0^\infty \frac{f(t)}{(t+z)^{r+1}}\, dt$.

The Stieltjes transform can be obtained by iteration of the Laplace transform.

Theorem 3.1. *Let* $W = \dfrac{f}{H^k} \in \mathcal{M}(r)$. *Then,* $\Lambda_z^r W = \dfrac{1}{\Gamma(r+1)} \displaystyle\int_0^\infty e^{-zt} t^r \widehat{W}(t)\, dt$, $\mathrm{Re}\,(z) > 0$, *where*

$$\widehat{W}(t) = t^k \widehat{f}(t) = t^k \int_0^\infty e^{-t\sigma} f(\sigma) d\sigma. \tag{3.4}$$

Proof.

$$\frac{1}{\Gamma(r+1)} \int_0^\infty e^{-zt} t^r \widehat{W}(t)\, dt = \frac{1}{\Gamma(r+1)} \int_0^\infty \int_0^\infty e^{-(z+\sigma)t}\, t^{r+k} f(\sigma) d\sigma\, dt \tag{3.5}$$

Because of the growth condition on f, the interchanging of the order of integration is justified.

Hence,

(3.6)

$$\begin{aligned}
\frac{1}{\Gamma(r+1)} \int_0^\infty \int_0^\infty e^{-(z+\sigma)t}\, t^{r+k} f(\sigma) d\sigma\, dt &= \frac{1}{\Gamma(r+1)} \int_0^\infty f(\sigma) \left(\int_0^\infty e^{-(z+\sigma)t}\, t^{r+k}\, dt\right) d\sigma \\
&= \frac{\Gamma(r+k+1)}{\Gamma(r+1)} \int_0^\infty \frac{f(\sigma)}{(\sigma+z)^{r+k+1}}\, d\sigma \\
&= \Lambda_z^r W,\ \ \mathrm{Re}\, z > 0.
\end{aligned}$$

Therefore, by (3.5) and (3.6),

$$\Lambda_z^r W = \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-zt} t^r \widehat{W}(t)\, dt,\ \ \mathrm{Re}\,(z) > 0\,.$$

Remark 3.2. *The Laplace transform operator (3.4) has similar properties as the classical Laplace transform (see* [1]*).*

The proofs of the following properties follow directly by using the previous theorem and the properties of the Laplace transform.

Properties. Let $W = \frac{f}{H^k} \in \mathcal{M}(r)$. Then for $r > -1$ and $z \in \mathbb{C}\backslash(-\infty, 0]$,

(1) $\Lambda_z^r \tau_c W = \Lambda_{z+c}^r W$, $c > 0$ and $\tau_c W = \frac{\tau_c f}{H^k}$, $\tau_c f(t) = f(t-c)$.
(2) $\Lambda_z^r D^m W = (r+1)_m \Lambda_z^{r+m} W$, $m = 1, 2, \ldots$
(3) $\frac{d^m}{dz^m} \Lambda_z^r W = (-1)^m (r+1)_m \Lambda_z^{r+m} W = (-1)^m \Lambda_z^r D^m W$, $m = 1, 2, \ldots$
(4) $\Lambda_z^{r+1}(tW) = \Lambda_z^r W - z\Lambda_z^{r+1} W$, where $tW = \frac{tf}{H^k} - \frac{kf}{H^{k-1}}$, $k \geq 2$.

Theorem 3.2. *Let $W \in \mathcal{M}(r)$. Then, there exist positive numbers α and β such that*

(i) $\Lambda_z^r W = o(z^{-\alpha})$ *as* $z \to 0$, $|arg\, z| \leq \psi < \frac{\pi}{2}$.
(ii) $\Lambda_z^r W = o(z^{-\beta})$ *as* $z \to \infty$, $|arg\, z| \leq \psi < \frac{\pi}{2}$.

Proof. Let $W = \frac{f}{H^k} \in \mathcal{M}(r)$, where for some positive constants M, α, and γ,

$$|f(t)| \leq M\, t^{r+k-\alpha}, \text{ for } t \geq \gamma.$$

(i) $\Lambda_z^r W = \frac{1}{\Gamma(r+1)} (t^{r+k} \widehat{f}(t))^\wedge(z)$, $\operatorname{Re} z > 0$.

Now,

$$\frac{t^{r+k} \widehat{f}(t)}{t^{r+k}} = \widehat{f}(t) \to 0 \text{ as } t \to \infty.$$

Therefore, by a classical Abelian theorem for the Laplace transform [3],

$$\frac{z^{r+k+1} (t^{r+k} \widehat{f}(t))^\wedge}{\Gamma(r+k+1)} \to 0 \text{ as } z \to 0, |arg z| \leq \psi < \frac{\pi}{2}.$$

Thus,

$$\lim_{\substack{z \to 0 \\ |arg z| \leq \psi < \frac{\pi}{2}}} z^{r+k+1} \Lambda_z^r W = 0.$$

This completes the proof of (i). Now, for the proof of (ii). There exist $A > 0$ and $B > 0$ such that

$$|t^{r+k} \widehat{f}(t)| \leq A t^{r+k} + \frac{B}{t^{1-\alpha}}, \; t > 0 \text{ (see [7], p. 211)}.$$

Thus, the function $t^{r+k} \widehat{f}(t)$ is locally integrable on $[0, \infty)$.

Now,

$$
\begin{aligned}
|\Lambda_z^r W| &\leq \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-t \operatorname{Re} z} \, t^{r+k} |\widehat{f}(t)| \, dt \\
&\leq \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-t \operatorname{Re} z} \left(A t^{r+k} + \frac{B}{t^{1-\alpha}} \right) dt \\
&= \frac{C}{(\operatorname{Re} z)^{r+k+1}} + \frac{D}{(\operatorname{Re} z)^{\alpha}}, \quad \operatorname{Re} z > 0,
\end{aligned}
$$

for some positive constants C, D.

Thus,

$$
\lim_{\substack{z \to \infty \\ |\arg z| \leq \psi < \frac{\pi}{2}}} z^{\beta} \Lambda_z^r W = 0, \text{ where } \beta = \frac{1}{2} \min\{\alpha, r+k+1\}.
$$

This completes the proof of the theorem.

4. Localization

Definition 4.1. *Let $W = \frac{f}{H^k} \in \mathcal{M}$. W is said to vanish on an open interval (a, b), denoted $W(t) = 0$ on (a, b), provided there exists a polynomial p with degree at most $k-1$ such that $p(t) = f(t)$ for $a < t < b$.*

The support of $W \in \mathcal{M}$, denoted supp W, is the complement of the largest open set on which W vanishes.

Remark 4.1.

(1) *The definition of W vanishing on an interval does not depend on the representation of W.*
(2) *The notion of an element of $\mathcal{M}$ vanishing on an interval is consistent with the notion of a function vanishing on an interval. That is, $f(t) = 0$ for $a < t < b$ if and only if $W_f(t) = 0$ on (a, b), where $f \in C_+(\mathbb{R})$ and $W_f = \frac{H * f}{H}$.*
(3) *It follows that if $W(t) = 0$ on (a, b), where $a < 0$, then $f(t) = 0$ for all $a < t < b$, where $W = \frac{f}{H^k}$.*

Example 4.1. *Recall $\delta = \frac{H^2}{H^2}$. Notice that $H^2(t) = t$ on the open interval $(0, \infty)$. Thus, $\delta(t) = 0$ on $(0, \infty)$. Also, $H(t) = 0$ on $(-\infty, 0)$. So, $\delta(t) = 0$ on $(-\infty, 0)$. Therefore, supp $\delta = \{0\}$.*

Example 4.2. *Let $W = \frac{f}{H^3}$, where* $f(t) = \begin{cases} t^2 & 0 \leq t < 2 \\ t+2 & t \geq 2 \\ 0 & t < 0. \end{cases}$

Then W has compact support. Notice that W vanishes on $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$, and hence, supp $W = \{0\} \cup \{2\}$.

Theorem 4.1. *Let $W \in \mathcal{M}$. If $DW(t) = 0$ on (a, b), then W is constant on (a, b).*

Proof. Let $W = \frac{f}{H^k}$ such that $DW = \frac{f}{H^{k+1}} = 0$ on (a, b). Therefore, there exists a polynomial $p(t) = \alpha_0 + \alpha_1 t + \ldots + \alpha_k t^k$ such that $f(t) = p(t)$, for all $a < t < b$. That is,

$$f - k!\alpha_k H^{k+1} = \alpha_0 H + \alpha_1 H^2 + \cdots + (k-1)!\alpha_{k-1} H^k \text{ on } (a, b).$$

Thus,

$$\frac{f}{H^k} - \frac{k!\alpha_k H^{k+1}}{H^k} = 0 \text{ on } (a, b).$$

That is, $W = \frac{f}{H^k} = k!\alpha_k H$ on (a, b). □

5. Abelian Theorems

Lemma 5.1. *Let $k \in \mathbb{N}$, $\alpha > 0$, and $r > -1$. If $f \in L^1_{loc}(\mathbb{R}^+)$ such that $f(t)t^{-r-k+\alpha}$ is bounded on $[b, \infty)$ (for some $b > 0$), then*
$\int_b^\infty \frac{f(t)}{(t+z)^{r+k+1}}\, dt$ is bounded in the half-plane $\operatorname{Re} z > 0$.

The following is an initial value theorem.

Theorem 5.1. *Let $W \in \mathcal{M}(r)$ and $\nu > -1$. If $W(t) \sim \xi t^\nu$ as $t \to 0^+$, then for $r > \nu$,*

$$\lim_{\substack{z \to 0 \\ |arg z| \le \psi < \frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

Proof. Since $W(t) \sim \xi t^\nu$ as $t \to 0^+$, $W(t) = W_g(t)$ on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, where $g \in L^1_{loc}(\mathbb{R}^+)$ and $\frac{g(t)}{t^\nu} \to \xi$ as $t \to 0^+$. We may assume that $g(t) = 0$ on $[\varepsilon, \infty)$.

Now, $W = W_g + V$, where for some $k \in \mathbb{N}$, $V \in \mathcal{M}_k(r)$ and $\operatorname{supp} V \subseteq [\varepsilon, \infty)$. Thus, by a classical Abelian theorem for the Stieltjes transform and the previous lemma, we obtain

$$\lim_{\substack{z \to 0 \\ |arg z| \le \psi < \frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r g}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi,$$

and,

$$\lim_{\substack{z \to 0 \\ |arg z| \le \psi < \frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r V}{\Gamma(r-\nu)\Gamma(\nu+1)} = 0.$$

Therefore,

$$\lim_{\substack{z \to 0 \\ |arg z| \le \psi < \frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

□

Lemma 5.2. *Let $a > 0$ and $k \in \mathbb{N}$. Then, for $n = 0, 1, 2, \ldots, k-1$ and $r > \nu > -1$,*

$$\lim_{\substack{z \to \infty \\ \operatorname{Re} z > 0}} z^{r-\nu} \int_a^\infty \frac{t^n}{(t+z)^{r+k+1}}\, dt = 0.$$

Proof. Follows by induction on k. □

Now, the final value theorem.

Theorem 5.2. *Let $W \in \mathcal{M}$ and $\nu > -1$. If $W(t) \sim \xi t^{\nu}$ as $t \to \infty$, then for $r > \nu$,*

$$\lim_{\substack{z\to\infty \\ |arg z|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

Proof. Since $W(t) \sim \xi t^{\nu}$ as $t \to \infty$, there exist $k \in \mathbb{N}$, $c > 0$, a polynomial p, and $g \in L^1_{loc}(\mathbb{R}^+)$ such that supp $g \subseteq [c, \infty)$, deg $p \le k-1$, and $f(t) = (H^k * g)(t) + p(t)$ on (c, ∞) with $\frac{g(t)}{t^{\nu}} \to \xi$ as $t \to \infty$.

It follows that $W \in \mathcal{M}_k(r)$ and that $W = W_g + V$, where $V = \frac{f-H^k*g}{H^k} \in \mathcal{M}_k(r)$ and supp $V \subseteq [0, c]$. By using a classical Abelian theorem and noting that $\Lambda_z^r W_g$ is the same as the classical Stieltjes transform of g, we obtain

$$\lim_{\substack{z\to\infty \\ |arg z|\le\psi<\frac{\pi}{2}}} \frac{z^{r-\nu}\Gamma(r+1)\Lambda_z^r W_g}{\Gamma(r-\nu)\Gamma(\nu+1)} = \xi.$$

Now, letting $T = f - H^k * g$, we obtain

$$\begin{aligned} z^{r-\nu}\Lambda_z^r V &= (r+1)_k\, z^{r-\nu}\int_0^{\infty}\frac{T(t)}{(t+z)^{r+k+1}}\,dt \\ &= (r+1)_k\, z^{r-\nu}\int_0^{c}\frac{T(t)}{(t+z)^{r+k+1}}\,dt + (r+1)_k\, z^{r-\nu}\int_c^{\infty}\frac{p(t)}{(t+z)^{r+k+1}}\,dt. \end{aligned}$$

By the previous lemma, for Re $z > 0$, it follows that the limit of the second term converges to zero as $z \to \infty$. Now, for Re $z > 0$,

$$\left| z^{r-\nu}\int_0^{c}\frac{T(t)}{(t+z)^{r+k+1}}\,dt \right| \le |z|^{-k-\nu-1}\int_0^{c}|T(t)|\,dt \to 0 \text{ as } z \to \infty.$$

The proof of the theorem is completed by observing that

$$z^{r-\nu}\Lambda_z^r W = z^{r-\nu}\Lambda_z^r W_g + z^{r-\nu}\Lambda_z^r V.$$

□

As a final remark, the map $\frac{f}{H^k} \to D^k f$ is a well-defined linear bijection from $\mathcal{M}(r)$ onto $J'(r)$, where D denotes the distributional differential operator [11] and

$$J'(r) = \{D^k f : k \in \mathbb{N}, f \in L^1_{loc}(\mathbb{R}^+), f(t)t^{-r-k+\alpha} \text{ bdd as } t \to \infty \text{ for some } \alpha > 0\}.$$

Moreover, the Stieltjes transform for $\frac{f}{H^k} \in \mathcal{M}(r)$ and the Stieltjes transform for $D^k f \in J'(r)$ are equivalent.

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3. Short or preliminary communication (original management paper of full format but of a smaller extent or of a preliminary character);
4. Scientific critique or forum (discussion on a particular scientific topic, based exclusively on management argumentation) and commentaries. Exceptionally, in particular areas, a scientific paper in the Journal can be in a form of a monograph or a critical edition of scientific data (historical, archival, lexicographic, bibliographic, data survey, etc.) which were unknown or hardly accessible for scientific research.

Professional articles:

1. Professional paper (contribution offering experience useful for improvement of professional practice but not necessarily based on scientific methods);
2. Informative contribution (editorial, commentary, etc.);
3. Review (of a book, software, case study, scientific event, etc.)

Language

The article should be in English. The grammar and style of the article should be of good quality. The systematized text should be without abbreviations (except standard ones). All measurements must be in SI units. The sequence of formulae is denoted in Arabic numerals in parentheses on the right-hand side.

Abstract and Summary

An abstract is a concise informative presentation of the article content for fast and accurate Evaluation of its relevance. It is both in the Editorial Office's and the author's best interest for an abstract to contain terms often used for indexing and article search. The abstract describes the purpose of the study and the methods, outlines the findings and state the conclusions. A 100- to 250-Word abstract should be placed between the title and the keywords with the body text to follow. Besides an abstract are advised to have a summary in English, at the end of the article, after the Reference list. The summary should be structured and long up to 1/10 of the article length (it is more extensive than the abstract).

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Acknowledgements

The name and the number of the project or programmed within which the article was realized is given in a separate note at the bottom of the first page together with the name of the institution which financially supported the project or programmed.

Tables and Illustrations

All the captions should be in the original language as well as in English, together with the texts in illustrations if possible. Tables are typed in the same style as the text and are denoted by numerals at the top. Photographs and drawings, placed appropriately in the text, should be clear, precise and suitable for reproduction. Drawings should be created in Word or Corel.

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Footnotes are given at the bottom of the page with the text they refer to. They can contain less relevant details, additional explanations or used sources (e.g. scientific material, manuals). They cannot replace the cited literature.
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Note