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Contents

Osgood type blow-up criterion for the 3D Boussinesq equations with partial viscosity

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A B S T R A C T

This paper is dedicated to studying the blow-up criterion of smooth solutions to the three dimensional Boussinesq equations with partial viscosity. By means of the Littlewood-Paley decomposition, we give an improved logarithmic Sobolev inequality and through this, we obtain the corre sponding blow-up criterion in a space larger than $\dot{B}^0_{\infty,\infty}$,which extends several previous works.

Keywords: Boussinesq equations; blow-up criterion; Besov space

1. Introduction

In this paper we consider the following Cauchy problem of 3D Boussinesq system:

$$\begin{cases} u_t - \nu\Delta u + u \cdot \nabla u + \nabla P = \theta e_3, \\ \theta_t - \kappa\Delta\theta + u \cdot \nabla\theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where, $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ is a vector field denoting the velocity, $\theta = \theta(x,t)$ is a scalar function denoting either the temperature in the content of thermal convection, or the density in the modeling of geophysical fluids, $P = P(x,t)$ the scalar pressure and $e_3 = (0,0,1)$ is the unit vector in the x_3 direction. The parameters $\nu, \kappa \geqslant 0$ represent the kinematic viscosity and molecular diffusion coefficients, respectively, while u_0 and b_0 are the given initial data.

The Boussinesq system has widely been used in atmospheric sciences and oceanic fluids[13, 18]. Local existence and uniqueness theories of solutions have been studied by many mathematicians and physicists(see,e.g., [1,2,14]).Chaeestablishedtheglobalregularitycriteriaforthe2DBoussinesq equationswithpartialdissipationincelebratedpaper[3].Recently,Ye[23]consideredthecasewith horizontaldissipationinthehorizontalvelocityequationandverticaldissipationinthetemperature equation. Similar resultsaboutglobal regularityfor2Dincompressiblefluidmodelspleaserefer to [11,12], andthereferencetherein. However, for the3DBoussinesqequations,whether theunique local strongsolutioncanexistsgloballyforgeneral large initialdata isanoutstandingchallenging openproblem.Therefore, itisimportanttostudythemechanismofblow-upandstructure of possible singularitiesofstrongorsmoothsolutionstothesystem(1.1). IshimuraandMorimoto[16]provedthe followingblow-upcriterion

$$\nabla u \in L^1(0,T;L^\infty(\mathbb{R}^3)). \tag{1.2}$$

FanandOzawa[9]andFanandZhou[10]establishedthefollowingrefinedblow-upcriterionfor system(1.1)as

$$\mathrm{curl}u \in L^1(0,T;\dot{B}^0_{\infty,\infty}), \tag{1.3}$$

with $\nu = 1, \kappa = 0$ (zero-diffusive case) and $\nu = 0, \kappa = 1$ (zero-viscosity case), respectively. By means of the Littlewood-Paley theory and Bony's paradifferential calculus, Qiu-Du-Yao [19] extended the condition to

$$u \in L^q\left(0,T;B^s_{p,\infty}(\mathbb{R}^3)\right), \frac{2}{q}+\frac{3}{p} = 1+s, \frac{3}{1+s} < p \leqslant \infty, -1 < s \leqslant 1, (p,s) \neq (\infty,1), \tag{1.4}$$

and further studied by Dong-Jia-Zhang[5] in the case of $\kappa = 0$. Zhang and Gala [24] gave an Osgood type regularity criterion for the Newton-Boussinesq equation, that is,

$$\sup_{q\geq 2}\int_0^T \frac{\|\bar{S}_q\nabla u(\tau)\|_{L^\infty}}{q\ln q}d\tau = \infty, \tag{1.5}$$

where $\bar{S}_q = \sum_{l=-q}^{q}\dot{\Delta}_l$, $\dot{\Delta}_l$ being the homogenous Fourier localization operator.

Recently, Wu-Hu-Liu [21] established the blow-up criterion (1.5) for the Boussinesq equation with full viscosities ($\nu = 1, \kappa = 1$) and Ren[20] obtained the following blow-up criterion

$$\int_0^T \sup_{2\leqslant q<\infty}\frac{\|\Delta_q \mathrm{curl}u(\tau)\|_{L^\infty}}{\log q}d\tau = \infty, \tag{1.6}$$

with the zero-viscosity constant. Here, Δ_q stands for nonhomegenous Littlewood-Paley projector operator. For more results about blow-up criterion for the system (1.1), we refer to [4, 6, 7, 15] and the reference therein.

Motivated by above-mentioned results, the purpose of this paper is to establish a blow-up criterion in a space (see definition 2.1) larger than $\dot{B}^0_{\infty,\infty}$. We should point out that, in the thesis [22], the author gave the blow-up criterion (1.3) for the fractional Boussinesq equations in n-dimensions($n \geq 2$) by using commutator operator estimate and the inequality

$$\|\nabla\theta(t)\|_{L^p} \leqslant \|\nabla\theta(T_0)\|_{L^p} exp\left\{\int_{T_0}^t \|\nabla u(\tau)\|_\infty\, d\tau\right\}, \quad p \in [2,\infty], \tag{1.7}$$

where L^∞-norm of the gradient of velocity can be controlled as

$$\|\nabla u\|_\infty \leq C\left(1 + \|u\|_{L^2} + \|\nabla\times u\|_{\dot{B}^0_{\infty,\infty}}\log\left(e + \|\Lambda^s u\|_{L^2}\right)\right), \quad s > 1+\frac{n}{2}.$$

However,forourcases,thelogarithmicSobolevinequalitybecomes

$$\|u\|_{L^\infty} \leqslant C\left(1 + \|u\|_{\dot{V}_\Theta}\ln(\|u\|_{H^m}+e)\ln(\ln(\|u\|_{H^m}+e)+e)\right), \quad m > \frac{3}{2}, \tag{1.8}$$

thenwecannotapplytheinequality(1.7)in[22].Therefore,somenewestimatesneedtobedeveloped. Inthispaper,wewill takeuseof the L^∞-norm of the temperature due to the special structure of the temperature equation. Through the boundness of $\|\theta\|_{L^\infty}$ and elaborate energy estimate which together the logarithmic Sobolev inequality, we obtain our main results. As a consequence, we improve the results in reference [9, 10, 20, 21].

The paper is organized as follows. In section 2, we recall the definition of Besov space and state our main results. The commutator operator estimate and the logarithmic Sobolev inequality are presentedin Lemma 2.1 and Lemma 2.2, respectively. Section 3 is devoted to proving Theorem 2.1.

Through this paper, C stands for some real positive constants which may be different in each occurrence.

2. Preliminaries and main results

Before presenting our results, we introduce some function spaces and some notations. First, we are going to recall some basic facts on Littlewood-Paley theory. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = 2\pi^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(\xi)dx.$$

Choose two nonnegative radial functions χ and φ, valued in the interval [0,1], supported in B ={$\xi \in \mathbb{R}^3$, $|\xi| \leqslant \frac{4}{3}\}$, $C = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$, respectively, such that

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3$$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$. The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} h(2^j y)u(x-y)dy,$$

and

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}(2^j y)u(x-y)dy.$$

Formally, $\dot{\Delta}_j$ is a frequency projection to the annulus$\{|\xi| \approx 2^j\}$, and $\dot{S}_j$ is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. By using of Littlewood-Paley's decomposition, we give the definition of the homogenous Besov space.

Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. $\mathcal{S}_h' = \{u \in \mathcal{S}'(\mathbb{R}^3); \lim_{j\to-\infty} \dot{S}_j = 0\}$ which can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$. Then the homogenous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{u \in \mathcal{S}_h'(\mathbb{R}^3); \|u\|_{\dot{B}^s_{p,q}} < \infty\}$$

where

$$\|u\|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j=-\infty}^{\infty} 2^{jsq} \|\dot{\Delta}_j u\|^q_{L^p})^{\frac{1}{q}}, & for \quad q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j u\|_{L^p}, & for \quad q = \infty. \end{cases}$$

Next, we introduce the space of Besov type, see [17].

Definition 2.1. *Let $\Theta(\alpha)(\geqslant 1)$ be a nondecreasing function on $[1, \infty]$. We denote by $\dot{V}_\Theta$ the set of tempered distributions u such that $\{u \in \mathcal{S}_h'(\mathbb{R}^3); \|u\|_{V_\Theta} < \infty\}$ and the norm is defined by*

$$\|u\|_{\dot{V}_\Theta} = \sup_{N=1,2\cdots} \frac{\| \sum_{j=-N}^N \dot{\Delta}_j u\|_{L^\infty}}{\Theta(N)}.$$

Remark 2.1. *We can easily see that $\|u\|_{\dot{V}_\Theta} \leqslant C\|u\|_{\dot{B}^0_{\infty,\infty}} \leqslant C\|u\|_{BMO} \leqslant C\|u\|_{L^\infty}$, provided $\Theta(N) \geqslant N$. In this paper, we will take $\Theta(N) = N\ln(N+e)$.*

Next, we present the following well-know commutator estimate and we can find the details in [8] for example.

Lemma 2.1. *Suppose that $s > 0$ and $p \in (1, \infty)$. Let f, g be two smooth functions such that $\nabla f \in L^{p_1}$, $\Lambda^s f \in L^{p_3}$, $\Lambda^{s-1} g \in L^{p_2}$ and $g \in L^{p_4}$, then there exist a constant C independent of f and g such that*

$$\|[\Lambda^s, f]g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}) \tag{2.1}$$

with $p_2, p_3 \in (1, \infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Here $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$.

Next, we give the following logarithmic Sobolev inequality which plays an important role in the proof of the blow-up criterion for the classic solutions.

Lemma 2.2. *Let $m > \frac{3}{2}$, then there exists constant C depending only on s, p and Θ such that*

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leqslant C(1 + \|u\|_{\dot{V}_\Theta} \Theta(\ln(\|u\|_{H^m(\mathbb{R}^3)} + e))) \tag{2.2}$$

for all $u \in H^m(\mathbb{R}^3)$.

Proof. By using Littlewood-Paley theory, we decompose the function into three parts. More precisely, we write

$$u(x) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u = u_l(x) + u_m(x) + u_h(x) \tag{2.3}$$

where

$$u_l(x) = \sum_{j<-N} \dot{\Delta}_j u, \quad u_m(x) = \sum_{-N\leqslant j\leqslant N} \dot{\Delta}_j u \quad \text{and} \quad u_m(x) = \sum_{j>N} \dot{\Delta}_j u. \tag{2.4}$$

For the low-frequency part $u_l(x)$, we can show that

$$\|u_l(x)\|_\infty \leqslant \sum_{j<-N} \|\dot{\Delta}_j u\|_\infty \leqslant \sum_{j<-N} C2^{3j/2}\|\dot{\Delta}_j u\|_{L^2} \leqslant C \sum_{j<-N} 2^{3j/2}\|u\|_{L^2} \leqslant C2^{-\frac{3}{2}N}\|u\|_{H^s}. \tag{2.5}$$

For the high-frequency part

$$\begin{aligned}\|u_h(x)\|_\infty &\leqslant \sum_{j>N} \|\dot{\Delta}_j u\|_\infty \\ &\leqslant \sum_{j>N} C2^{3j/2}\|\dot{\Delta}_j u\|_{L^2} \\ &= C\sum_{j>N} 2^{sj}\|\dot{\Delta}_j u\|_{L^2} 2^{-sj+3j/2} \\ &\leqslant C2^{-(s-3/2)N}\|u\|_{H^s},\end{aligned} \tag{2.6}$$

for $s > \frac{3}{2}$, where we have used the following Bernstein estimate

$$\|\dot{\Delta}_j u\|_{L^{p_2}} \leqslant C2^{jd(\frac{1}{p_1}-\frac{1}{p_2})}\|\dot{\Delta}_j u\|_{L^{p_1}}, \quad \text{for} \quad 1 \leqslant p_1 \leqslant p_2 \leqslant \infty,$$

and the norm equivalent between $\|\cdot\|_{\dot{H}^s}$ and $\|\cdot\|_{\dot{B}^s_{2,2}}$.

Next, we consider $u_m(x)$, by definition 2.1 we have

$$\|u_m(x)\|_\infty \leqslant \sum_{j=-N}^{N} \|\dot{\Delta}_j u\|_\infty \leqslant \Theta(N)\|u\|_{\dot{V}_\Theta}. \tag{2.7}$$

Taking from (2.3) to (2.7) into consideration, we get

$$\|u(x)\|_\infty \leqslant C(2^{-\frac{3}{2}N}\|u\|_{H^s} + 2^{-(s-3/2)N}\|u\|_{H^s} + \Theta(N)\|u\|_{\dot{V}_\Theta}). \tag{2.8}$$

If we take $N = [\frac{\ln(\|u\|_{H^s}+e)}{min(s-3/2,3/2)\ln 2}] + 1$, where $[\cdot]$ denotes Gausss symbol, then we have the desired estimate (2.2). □

Now, we state our main results.

Theorem 2.1. *Suppose $\nu > 0$, $\kappa = 0$ with $(u_0, \theta_0) \in H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. Let $T > 0$ be the maximum time such that (u, θ) be a local smooth solution to the system (1.1). If $T < \infty$, then*

$$\limsup_{t\nearrow T}(\|u(t)\|_{H^s} + \|\theta(t)\|_{H^s}) = \infty, \tag{2.9}$$

if and only if

$$\int_0^T \|\nabla u\|_{\dot{V}_\Theta} d\tau = \infty. \tag{2.10}$$

Remark 2.2. *For the zero-diffusion case, that is, $\nu > 0$, $\kappa = 0$, the stiuation becomes more difficult. The main obstacle is that the temperature $\theta(x,t)$ in the transport equation does not gain any smoothness. Hence, the blow-up issue of the zero-diffusive Boussinesq equations is more difficult than that of full viscous Boussinesq system. The result (2.10) is an improvement of (1.5) in [21].*

Remark 2.3. *After some slight modifications, our methods can also be applied to the zero-viscosity case, that is, $\nu = 0, \kappa = 1$. More precisely, J_2 and K_2 in (3.10) and (3.15) can be estimated as*

$$\begin{aligned} J_2 &\leqslant C\left(\|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{L^4}\|\nabla^3\theta\|_{L^2}\right) \\ &\leqslant \|\nabla^3\theta\|_{L^2}^2 + C\left(\|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2 + \|\nabla u\|_{L^\infty}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\theta\|_{L^\infty}\|\nabla^2\theta\|_{L^2}\right) \\ &\leqslant \|\nabla^3\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\left(\|\nabla^2\theta\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + 1\right), \end{aligned} \tag{2.11}$$

similarily

$$\begin{aligned} K_2 &\leqslant C\left(\|\nabla^3 u\|_{L^2}\|\nabla\theta\|_{L^3}\|\nabla^3\theta\|_{L^6} + \|\nabla^2 u\|_{L^2}\|\nabla^2\theta\|_{L^3}\|\nabla^3\theta\|_{L^6} + \|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2\right) \\ &\leqslant \|\nabla^4\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\left(\|\nabla^2\theta\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + 1\right). \end{aligned} \tag{2.12}$$

Remark 2.4. *From the Biot-Savart law, we have $\|\nabla u\|_{\dot V_\Theta} \leqslant C\|\nabla u\|_{\dot B^0_{\infty,\infty}} \leqslant C\|w\|_{\dot B^0_{\infty,\infty}}$, which refined the results in [9, 10].*

Remark 2.5. *For the nonhomogenous case, our space becomes $V_\Theta = \{u \in \mathcal{S}'(\mathbb{R}^3); \|u\|_{V_\Theta} < \infty\}$, where*

$$\|u\|_{V_\Theta} = \sup_{N\geq 2} \frac{\left\|\sum_{j=-1}^N \Delta_j u\right\|_{L^\infty}}{\Theta(N)}.$$

Then our results is an improvement of (1.6) in [20], since

$$\sup_{N\geq 2} \frac{\left\|\sum_{j=-1}^N \Delta_j u\right\|_{L^\infty}}{N\ln(N+e)} \leqslant C \sup_{2\leqslant q<\infty} \frac{\|\Delta_q u\|_{L^\infty}}{\ln q} \leqslant C\|u\|_{B^0_{\infty,\infty}}.$$

Moreover, we give the following function for example

$$f(x) = \log\left(\frac{1}{|x|} + e\right)\log\log\left(\frac{1}{|x|} + e\right) \in V_\Theta,$$

but does not belongs to $B^0_{\infty,\infty}$, please refer to [17, 20] for details.

3. Proof of theorem 2.1

Proof. We consider the 3D Boussinesq equations (1.1) with $\nu > 0$, $\kappa = 0$. It is easy to see that (2.10) implies (2.9), since $\|\nabla u\|_{\dot V_\Theta} \leqslant C\|\nabla u\|_\infty \leqslant C\|u(t)\|_{H^3}$. Hence, we only need to prove that (2.9) implies (2.10). If (2.10) is false, then there exists a constant C such that

$$\int_0^T \|\nabla u\|_{\dot V_\Theta} d\tau < C. \tag{3.1}$$

Multiplying the second equation of (1.1) by u and θ, respectively, integrating and using the divergence-free condition $\nabla \cdot u = 0$, we immediately have

$$\|u(t)\|_{L^2} \leqslant C\left(\|u_0\|_{L^2} + \|\theta_0\|_{L^2}\right), \quad \|\theta(t)\|_{L^2} \leqslant \|\theta_0\|_{L^2}, \tag{3.2}$$

for any $t \in [0, T]$. Furthermore, we have $\|\theta(t)\|_{L^p} \leqslant \|\theta_0\|_{L^p}$, for $p \in [2, \infty]$.

Next, we are going to give H^1, H^2 and H^3 estimates to complete our proof.

(H^1 estimate). Multiplying the first two equations of (1.1) by Δu and $\Delta\theta$, respectively, after integrating by parts and taking the divergence-free property into account, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \, dx + \int_{\mathbb{R}^3} (u \cdot \nabla\theta) \cdot \Delta\theta \, dx - \int_{\mathbb{R}^3} (\theta e_n) \cdot \Delta u \, dx \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{3.3}$$

After integrating by parts several times, one can conclude that the three terms above can be bounded as

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \, dx \\
&= -\int_{\mathbb{R}^3} \partial_k(u_i \partial_i u_j)\partial_k u_j \, dx \\
&= -\int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j + u_i \partial_i \partial_k u_j \partial_k u_j \, dx \\
&\leqslant C\|\nabla u\|_\infty \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.4}$$

Similarly, we have

$$I_2 = \int_{\mathbb{R}^3} (u \cdot \nabla\theta) \cdot \Delta\theta \, dx \leqslant C\|\nabla u\|_\infty \|\nabla\theta\|_{L^2}^2, \tag{3.5}$$

and for the third term, we can show as

$$I_3 = -\int_{\mathbb{R}^3} (\theta e_n) \cdot \Delta u \, dx \leqslant \|\nabla\theta\|_{L^2}\|\nabla u\|_{L^2} \leqslant \|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2. \tag{3.6}$$

Plugging the estimates (3.4), (3.5), (3.6) into (3.3), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla\theta(t)\|_{L^2}^2) + \|\nabla^2 u(t)\|_{L^2}^2 \\
&\leqslant C\|\nabla u\|_\infty \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_\infty \|\nabla\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \\
&\leqslant C(1 + \|\nabla u\|_\infty)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned} \tag{3.7}$$

(H^2 estimate) applying ∇^2 to the first two equations of (1.1), respectively, integrating and adding the resulting equations together it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2) + \|\nabla^3 u(t)\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \nabla^2(u \cdot \nabla u) \cdot \nabla^2 u \, dx - \int_{\mathbb{R}^3} \nabla^2(u \cdot \nabla\theta) \cdot \nabla^2\theta \, dx + \int_{\mathbb{R}^3} \nabla^2(\theta e_n) \cdot \nabla^2 u \, dx \\
&:= J_1 + J_2 + J_3.
\end{aligned} \tag{3.8}$$

In what follows, we will deal with each term on the right-hand side of (3.8) separately below

$$J_1 = -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla u)\cdot\nabla^2 u\,dx = -\int_{\mathbb{R}^3}[\nabla^2, u\cdot\nabla]u\cdot\nabla^2 u\,dx$$
$$\leqslant \|[\nabla^2, u\cdot\nabla]u\|_{L^2}\|\nabla^2 u\|_{L^2}$$
$$\leqslant C\|\nabla u\|_\infty\|\nabla^2 u\|_{L^2}^2, \tag{3.9}$$

where we have used the Lemma 2.1 commutator estimate.

$$J_2 = -\int_{\mathbb{R}^3}\nabla^2(u\cdot\nabla\theta)\cdot\nabla^2\theta\,dx$$
$$\leqslant \|\nabla^3 u\|_{L^2}\|\nabla\theta\|_{L^4}^2 + \|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2$$
$$\leqslant C\left(\|\nabla^3 u\|_{L^2}\|\theta\|_{L^\infty}\|\nabla^2\theta\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^2}^2\right)$$
$$\leqslant \|\nabla^3 u\|_{L^2}^2 + C\left(1+\|\nabla u\|_\infty\right)\|\nabla^2\theta\|_{L^2}, \tag{3.10}$$

where we have used the following Gagliardo-Nirenberg inequality

$$\|\nabla\theta\|_{L^4}^2 \leqslant C\|\theta\|_{L^\infty}\|\nabla^2\theta\|_{L^2}.$$

The third term can be estimated as

$$J_3 = \int_{\mathbb{R}^3}\nabla^2(\theta e_n)\cdot\nabla^2 u\,dx \leqslant \|\nabla^2\theta\|_{L^2}\|\nabla^2 u\|_{L^2}. \tag{3.11}$$

$$\frac{d}{dt}(\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2\theta(t)\|_{L^2}^2) \leqslant C\left(1+\|\nabla u\|_\infty)(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2\right). \tag{3.12}$$

(H^3 estimate) applying ∇^3 to the first two equations of (1.1), respectively, integrating and adding the resulting equations together, it follows that

$$\frac{1}{2}\frac{d}{dt}(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3\theta(t)\|_{L^2}^2) + \|\nabla^4 u(t)\|_{L^2}^2$$
$$= -\int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla u)\cdot\nabla^3 u\,dx - \int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla b)\cdot\nabla^3 b\,dx + \int_{\mathbb{R}^3}\nabla^3(\theta e_n)\cdot\nabla^3 u\,dx$$
$$:= K_1 + K_2 + K_3. \tag{3.13}$$

$K_i (i = 1, 2, 3)$ can be bounded as

$$K_1 = -\int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla u)\cdot\nabla^3 u\,dx = -\int_{\mathbb{R}^3}[\nabla^3, u\cdot\nabla]u\cdot\nabla^3 u\,dx$$
$$\leqslant \|[\nabla^3, u\cdot\nabla]u\|_{L^2}\|\nabla^3 u\|_{L^2}$$
$$\leqslant C\|\nabla u\|_\infty\|\nabla^3 u\|_{L^2}^2, \tag{3.14}$$

and

$$K_2 = -\int_{\mathbb{R}^3}\nabla^3(u\cdot\nabla\theta)\cdot\nabla^3\theta\,dx$$
$$\leqslant C\left(\|\nabla^4 u\|_{L^2}\|\nabla\theta\|_{L^4}\|\nabla^2\theta\|_{L^4} + \|\nabla^3 u\|_{L^6}\|\nabla^2\theta\|_{L^2}\|\nabla^2\theta\|_{L^3} + \|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2\right)$$

$$\leqslant C\left(\|\nabla^4 u\|_{L^2}\|\theta\|_{L^\infty}\|\nabla^3\theta\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2\right)$$
$$\leqslant \|\nabla^4 u\|_{L^2}^2+C\left(1+\|\nabla u\|_\infty\right)\|\nabla^3\theta\|_{L^2}, \tag{3.15}$$

where we have used Young's inequality and the following Gagliardo-Nirenberg inequality in the three dimension

$$\|\nabla\theta\|_{L^4}\leqslant C\|\theta\|_{L^\infty}^{\frac{5}{6}}\|\nabla^3\theta\|_{L^2}^{\frac{1}{6}},\quad \|\nabla^2\theta\|_{L^4}\leqslant C\|\theta\|_{L^\infty}^{\frac{1}{6}}\|\nabla^3\theta\|_{L^2}^{\frac{5}{6}},$$
$$\|\nabla^2\theta\|_{L^2}\leqslant C\|\theta\|_{L^\infty}^{\frac{2}{3}}\|\nabla^3\theta\|_{L^2}^{\frac{1}{3}},\quad \|\nabla^2\theta\|_{L^3}\leqslant C\|\theta\|_{L^\infty}^{\frac{1}{3}}\|\nabla^3\theta\|_{L^2}^{\frac{2}{3}},$$

and

$$\|\nabla^3 u\|_{L^6}\leqslant C\|\nabla^4 u\|_{L^2}.$$

The third term can be estimated as

$$K_3=\int_{\mathbb{R}^3}\nabla^3(\theta e_n)\cdot\nabla^3 u\,dx\leqslant\|\nabla^3\theta\|_{L^2}\|\nabla^3 u\|_{L^2}. \tag{3.16}$$

Combining the above estimates into (3.13), we get

$$\frac{d}{dt}(\|\nabla^3 u(t)\|_{L^2}^2+\|\nabla^3\theta(t)\|_{L^2}^2)\leqslant C\Big(1+\|\nabla u\|_\infty)(\|\nabla^3 u\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2\Big). \tag{3.17}$$

Taking (3.2), (3.7), (3.12), (3.17) into account and adding them up, integrating the resulting inequality from 0 to T, which together with Lemma 2.2, we can infer that

$$\ln\left(M(T)+e\right)\leqslant\ln\left(M(0)+e\right)+C\int_0^T(1+\|\nabla u\|_{\dot V_\Theta})\ln\ln(M(\tau)+e)\ln(M(\tau)+e)d\tau, \tag{3.18}$$

where $M(t):=\max\limits_{\tau\in[0,t]}(\|u(\tau)\|_{H^3}^2+\|b(\tau)\|_{H^3}^2)$ for any $t\in[0,T]$. Then, we take use of Gronwall's inequality, finally we have

$$\ln\left(M(T)+e\right)\leqslant\exp\left\{\exp C\int_0^T\|\nabla u\|_{\dot V_\Theta}\,d\tau\right\}.$$

Therefore, we get the boundness of $H^3\times H^3$-norm of (u,θ) for all $t\in[0,T]$ which leads a contradiction, this completes the proof of Theorem 2.1. □

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Conflict of Interest

The author declare no conflicts of interest in this paper.

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Results on spirallike *p*-valent functions

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ABSTRACT

In this paper, we introduce two new subclasses of p-valent spirallike functions of order α. We prove necessary and sufficient conditions for these newly defined classes and also point out someknown consequences of our results.

Keywords: spirallike function; p-valent function; necessary and sufficient conditions

1. Introduction

Let $A(p)$ denote the class of all functions f defined b

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in N = 1, 2, 3...) \tag{1.1}$$

which are analytic and p-valent in the unit disk

$$\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}.$$

For a real number α $(0 \leq \alpha < p)$ the well-known subclasses $S^*_p(\alpha)$, p-valently starlike functions of order α and $C_p(\alpha)$, p-valently convex functions of order α of $A(p)$ are given by

$$S^*(\alpha, p) = \left\{ f \in A(p) : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{E}) \right\},$$
$$C(\alpha, p) = \left\{ f \in A(p) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{E}) \right\}.$$

For $|\beta| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, a function $f \in A$ is said to be β-spirallike of order α in $\mathbb{E}$ if

$$\Re\left\{ e^{\iota\beta} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos\beta \quad (z \in \mathbb{E}). \tag{1.2}$$

The class of all such functions is denoted by $S_\beta(\alpha)$ [3], (also see [5, 14, 15]). In recent years many interesting subclasses of analytic univalent, multivalent and spirallike functions and their many special cases were investigated, see for example [1, 2, 6, 7, 8, 9, 10].

Motivated and inspired by the above mention work, we here define the following.

Definition 1.1. A function $f \in A(p)$ belongs to the class $S_\beta(\alpha, p)$ if it satisfies the inequality

$$\left| \frac{f^{(p-1)}(z)}{e^{\iota\beta} z f^{(p)}(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \qquad z \in \mathbb{E},$$

for some real β and $0 < \alpha < 1$, where $f^{(p)}(z)$ is the p^{th} derivative of $f(z)$.

Remark 1.2. First of all, it is easily seen that, for

$$p = 1 \quad \text{and} \quad \beta = 0, S_0(\alpha, 1) = M(\alpha),$$

where $M(\alpha)$ is a function class introduced and studied in [12]. Secondly, we have

$$p = 1, \quad S_\beta(\alpha, 1) = S_\beta(\alpha),$$

where $S_\beta(\alpha)$ is a function class introduced by Owa and Kamali [13].

Definition 1.3. A function $f \in A(p)$ is said to be in the class $C_\beta(\alpha, p)$ if it satisfies the inequality

$$\left| \frac{f^{(p)}(z)}{e^{\iota\beta} \left(z f^{(p)}(z)\right)'} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \qquad z \in \mathbb{E}, \tag{1.3}$$

for some real β and $0 < \alpha < 1$, where $f^{(p)}(z)$ is the pth derivative of $f(z)$.

As a special case, the class $C_\beta(\alpha, 1) = K_\beta(\alpha)$, is introduced by Owa and Kamali [13]. Using essentially their technique, we prove the main results for the classes $S_\beta(\alpha, p)$ and $C_\beta(\alpha, p)$ which is the main motivation of this paper.

2. Preliminary results

Lemma 2.1. *[4]. Let $\phi(u, \upsilon)$ be a complex-valued function such that*

$$\phi : D \to \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C}$$

$\mathbb{C}$ being the complex plane and let $u = u_1 + \iota u_2$ and $v = v_1 + \iota v_2$. Suppose that the function $\phi(u, \upsilon)$ satisfies each of the following conditions

1. *$\phi(u, \upsilon)$ is continuous in D;*
2. *$(1, 0) \in D$ and $\Re\{\phi(1, 0)\} > 0$;*
3. *$\Re\{\phi(\iota u_2, v_1)\} \leq 0$ for all $(\iota u_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$. Let*

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

be analytic (regular) in the unit disk $\mathbb{E}$ such that

$$\left(p(z), zp'(z)\right) \in D, \quad \text{for all } z \in \mathbb{E}.$$

If

$$\Re\left\{\phi\left(p(z), zp'(z)\right)\right\} > 0, \quad \text{then} \quad \Re\{p(z)\} > 0 \;\; (z \in \mathbb{E}).$$

3. Main results

Theorem 3.1. *A function $f \in S_\beta(\alpha, p)$ if and only if, $\Re\left(e^{\iota\beta} \frac{z f^{(p)}(z)}{f^{(p-1)}(z)}\right) > \alpha$.*

Proof. Let $f(z) \in S_\beta(\alpha, p)$, then we can write

$$\left|\frac{1}{e^{\iota\beta}F(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \ \ (z \in E)$$

where $F(z) = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$. From above, we have

$$\left|\frac{2\alpha - e^{\iota\beta}F(z)}{2\alpha e^{\iota\beta}F(z)}\right| < \left(\frac{1}{2\alpha}\right)$$

$$\begin{aligned}
&\Leftrightarrow \left|2\alpha - e^{\iota\beta}F(z)\right|^2 < \left(e^{\iota\beta}F(z)\right)^2 \\
&\Leftrightarrow \left[2\alpha - e^{\iota\beta}F(z)\right]\left[\overline{2\alpha - e^{\iota\beta}F(z)}\right] < \left(e^{\iota\beta}F(z)\right)\left[\overline{e^{\iota\beta}F(z)}\right] \\
&\Leftrightarrow \left[2\alpha - e^{\iota\beta}F(z)\right]\left[2\alpha - e^{-\iota\beta}\overline{F(z)}\right] < \left(e^{\iota\beta}F(z)\right)\left[e^{-\iota\beta}\overline{F(z)}\right] \\
&\Leftrightarrow 4\alpha^2 - 2\alpha e^{-\iota\beta}\overline{F(z)} - 2\alpha e^{\iota\beta}F(z) + F(z)\overline{F(z)} < F(z)\overline{F(z)} \\
&\Leftrightarrow 4\alpha^2 - 2\alpha\left(e^{-\iota\beta}\overline{F(z)} + e^{\iota\beta}F(z)\right) < 0 \\
&\Leftrightarrow 2\alpha - 2\Re(e^{\iota\beta}F(z)) < 0 \\
&\Leftrightarrow -2\Re(e^{\iota\beta}F(z)) < -2\alpha \\
&\Leftrightarrow \Re\left(e^{\iota\beta}\frac{zf^{(P)}(z)}{f^{(P-1)}(z)}\right) > \alpha.
\end{aligned}$$

This complete the proof. □

When $p = 1$ we have the following known result proved by Owa and Kamali [13].

Corollary 3.2. $f(z) \in S_\beta(\alpha)$ *iff* $\Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha$.

Theorem 3.3. *If* $f(z) \in A(p)$ *satisfies*

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!}\left(n + 1 + \left|(n+1) - 2\alpha e^{-\iota\beta}\right|\right)\left|a_{n+p}\right| \leq 1 - \left|1 - 2\alpha e^{-\iota\beta}\right| \tag{3.1}$$

for some $|\beta| < \frac{\pi}{2}$ *and* $0 < \alpha < \cos\beta$, *then* $f(z) \in S_\beta(\alpha, p)$.

Proof. If $f(z) \in S_\beta(\alpha, p)$ then, it suffices to show that

$$\left|\frac{2\alpha - e^{\iota\beta}F(z)}{e^{\iota\beta}F(z)}\right| < 1$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, where $F(z) = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$.

Now we have

$$\left|\frac{2\alpha - e^{\iota\beta}F(z)}{e^{\iota\beta}F(z)}\right| = \left|\frac{2\alpha e^{-\iota\beta}f^{(p-1)}(z) - zf^{(p)}(z)}{zf^{(p)}(z)}\right|$$

$$= \left| \frac{2\alpha e^{-\iota\beta} - 1 + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left(2\alpha e^{-\iota\beta} - (n+1)\right) a_{n+p} z^n}{1 + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} (n+1) a_{n+p} z^n} \right|$$

$$\leq \frac{\left|2\alpha e^{-\iota\beta} - 1\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left|\left(2\alpha e^{-\iota\beta} - (n+1)\right)\right| \left|a_{n+p}\right| |z^n|}{1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} (n+1) \left|a_{n+p}\right| |z^n|}$$

$$< \frac{\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left|(n+1) - 2\alpha e^{-\iota\beta}\right| \left|a_{n+p}\right|}{1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} (n+1) \left|a_{n+p}\right|}. \tag{3.2}$$

The last expression in (3.2) is bounded above by 1 if

$$\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left|(n+1) - 2\alpha e^{-\iota\beta}\right| \left|a_{n+p}\right| \leq 1 - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left|(n+1) a_{n+p}\right|. \tag{3.3}$$

After simplification of (3.3) we have

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left\{\left(n + 1 + \left|(n+1) - 2\alpha e^{-\iota\beta}\right|\right)\right\} \left|a_{n+p}\right| \leq 1 - \left|1 - 2\alpha e^{-\iota\beta}\right|.$$

Therefore, $f(z) \in S_\beta(\alpha, p)$ for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$. □

When $\beta = 0$ and $p = 1$, we have the following result proved by Owa *et al.* [12].

Corollary 3.4. *Let* $0 < \alpha < 1$. *If* $f(z) \in A$ *satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq \frac{1}{2} (1 - |1 - 2\alpha|) = \begin{cases} \alpha; & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1 - \alpha; & \left(\frac{1}{2} < \alpha < 1\right) \end{cases}$$

then $f(z) \in M(\alpha)$.

Taking $\beta = \frac{\pi}{4}$ in above Theorem we have the following result.

Corollary 3.5. *If* $f(z) \in A(p)$ *satisfies*

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} \left(n + 1 + \sqrt{(n+1)^2 - 2\sqrt{2}\alpha(n+1) + 4\alpha^2}\right) \left|a_{n+p}\right| \leq 1 - \sqrt{1 - 2\sqrt{2}\alpha + 4\alpha^2}$$

for some $0 < \alpha < \frac{\sqrt{2}}{2}$, *then* $f(z) \in S_{\frac{\pi}{4}}(\alpha, p)$.

Theorem 3.6. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S_\beta(\alpha, p)$ *and let*

$$0 < \lambda \leq \frac{1}{2(\cos\beta - \alpha)}, \quad 0 < \alpha < \cos\beta. \tag{3.4}$$

Then we have

$$\Re\left\{\left(\frac{f^{(p-1)}(z)}{z}\right)^{\lambda e^{\iota\beta}}\right\} > \frac{(p!)^{-\lambda e^{\iota\beta}}}{2\lambda(\cos\beta - \alpha) + 1} \; (z \in E). \tag{3.5}$$

Proof. If we put

$$A = \frac{1}{2\lambda(\cos\beta - \alpha) + 1}$$

and

$$\left(\frac{f^{(p-1)}(z)}{p!z}\right)^{\lambda e^{\iota\beta}} = (1-A)p(z) + A \tag{3.6}$$

where λ satisfies (3.4) then $p(z)$ is regular in the unit disk E and $p(z) = 1 + p_1 z + p_2 z^2 + ...$
Logarithmic differentiation of (3.6) yields

$$\lambda e^{\iota\beta}\left[\frac{f^{(p)}(z)}{f^{(p-1)}(z)} - \frac{1}{z}\right] = (1-A)\frac{p'(z)}{(1-A)p(z)+A}.$$

This can be written as

$$e^{\iota\beta}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - e^{\iota\beta} = (1-A)\frac{zp'(z)}{\lambda\{(1-A)p(z)+A\}},$$

equivalently

$$e^{\iota\beta}\frac{zf^{(p)}(z)}{f^{p-1}(z)} - \alpha = e^{\iota\beta} - \alpha + (1-A)\frac{zp'(z)}{\lambda\{(1-A)p(z)+A\}}. \tag{3.7}$$

Since $f(z) \in S_\beta(\alpha, p)$ then from (3.7) we have

$$\Re\left\{e^{\iota\beta} - \alpha + (1-A)\frac{zp'(z)}{\lambda\{(1-A)p(z)+A\}}\right\} > 0, \quad (z \in E, 0 < \alpha < \cos\beta).$$

Let us consider the functional $\theta(u, v)$ defined by

$$\theta(u, v) = e^{\iota\beta} - \alpha + (1-A)\frac{v}{\lambda\{(1-A)u+A\}},$$

where $u = p(z)$ and $v = zp'(z)$. Then $\theta(u, v)$ is continuous in $D = \left(\mathbb{C} - \left\{\frac{A}{A-1}\right\}\right) \times \mathbb{C}$.
Also, $(1, 0) \in D$ and $\Re\{\theta(1, 0)\} = \cos\beta - \alpha > 0$. Furthermore, for all $(\iota u_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$, we have

$$\begin{aligned}
\Re\{\theta(\iota u_2, v_1)\} &= \cos\beta - \alpha + \Re\left\{(1-A)\frac{v_1}{[\lambda(1-A)\iota u_2 + A]}\right\} \\
&= \cos\beta - \alpha + \frac{A(1-A)v_1}{\lambda\left[(1-A)^2u_2^2 + A^2\right]} \\
&< \cos\beta - \alpha - \frac{A(1-A)\left(1+u_2^2\right)}{2\lambda\left[(1-A)^2u_2^2 + A^2\right]} \\
&= (\cos\beta - \alpha)\frac{A^2\left[4\lambda^2(\cos\beta-\alpha)^2 - 1\right]u_2^2}{\left[(1-A)^2u_2^2 + A^2\right]} \\
&\leq 0,
\end{aligned}$$

because $0 < \alpha < \cos\beta$ and $4\lambda^2(\cos\beta - \alpha)^2 - 1 \leq 0$ implies that $\lambda \leq \frac{1}{2(\cos\beta-\alpha)}$.
Therefore, the functional $\theta(u, v)$ satisfies all the conditions of Lemma 2.1.This proves that $\Re\{p(z)\} > 0$, that is from (3.6)

$$\begin{aligned}\Re\left(\frac{f^{(p-1)}(z)}{p!z}\right)^{\lambda e^{\iota\beta}} &> A\\ \Re\left(\frac{f^{(p-1)}(z)}{z}\right)^{\lambda e^{\iota\beta}} &> \frac{(p!)^{-\lambda e^{\iota\beta}}}{2\lambda(\cos\beta-\alpha)+1}.\end{aligned}$$

This completes the proof. □

For $\beta = 0$ and $p = 1$, in above theorem we have the following known result given by [11].

Corollary 3.7. *Let $f \in A$ be in the class $S_0(\alpha, 1)$ and $0 < \lambda \leq \frac{1}{2(1-\alpha)}$, $0 < \alpha < 0$ then*

$$\Re\left(\frac{f(z)}{z}\right)^{\lambda} > \frac{1}{2\lambda(1-\alpha)+1}, \quad z \in E.$$

Theorem 3.8. *A function $f \in C_\beta(\alpha, p)$ if and only if*

$$\Re\left\{e^{\iota\beta}\left(1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\} > \alpha.$$

Proof. Let $f(z) \in C_\beta(\alpha, p)$, then we can write

$$\left|\frac{1}{e^{\iota\beta}G(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}.$$

This can be written as

$$\begin{aligned}\left|\frac{1}{e^{\iota\beta}G(z)} - \frac{1}{2\alpha}\right| &< \frac{1}{2\alpha} \Leftrightarrow \left|\frac{2\alpha - e^{\iota\beta}G(z)}{2\alpha e^{\iota\beta}G(z)}\right| < \frac{1}{2\alpha}\\ &\Leftrightarrow \left|2\alpha - e^{\iota\beta}G(z)\right|^2 < \left(e^{\iota\beta}G(z)\right)^2\\ &\Leftrightarrow \left(2\alpha - e^{\iota\beta}G(z)\right)\overline{(2\alpha - e^{\iota\beta}G(z))} < \left(e^{\iota\beta}G(z)\right)\overline{(e^{\iota\beta}G(z))}\\ &\Leftrightarrow \left(2\alpha - e^{\iota\beta}G(z)\right)\left(2\alpha - e^{-\iota\beta}\overline{G(z)}\right) < \left(e^{\iota\beta}G(z)\right)\left(e^{-\iota\beta}\overline{G(z)}\right)\\ &\Leftrightarrow 4\alpha^2 - 2\alpha\left[e^{-\iota\beta}\overline{G(z)} + e^{\iota\beta}G(z)\right] + G(z)\overline{G(z)} < G(z)\overline{G(z)}\\ &\Leftrightarrow 4\alpha^2 - 2\alpha\left[e^{-\iota\beta}\overline{G(z)} + e^{\iota\beta}G(z)\right] < 0\\ &\Leftrightarrow 2\alpha\left[2\alpha - \Re\left(e^{\iota\beta}G(z)\right)\right] < 0\\ &\Leftrightarrow 2\alpha - 2\Re\left(e^{\iota\beta}G(z)\right) < 0\\ &\Leftrightarrow \Re\left\{e^{\iota\beta}\left(1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right)\right\} > \alpha.\end{aligned}$$

This completes the proof. □

Theorem 3.9. *If $f(z) \in A(p)$ satisfies*

$$\sum_{n=1}^{\infty}\frac{(p+n)!}{n!}\left\{n+1+\left|(n+1)-2\alpha e^{-\iota\beta}\right|\right\}\left|a_{n+p}\right| \leq 1 - \left|1-2\alpha e^{-\iota\beta}\right| \tag{3.8}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in C_\beta(\alpha, p)$.

Proof. To prove that $f(z) \in C_\beta(\alpha, p)$ we need to prove that

$$\left|\frac{2\alpha - e^{\iota\beta}G(z)}{e^{\iota\beta}G(z)}\right| < 1 \tag{3.9}$$

for some $|\beta| < \frac{\pi}{2}, 0 < \alpha < 1$, where $G(z) = 1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}$.
For this consider the left hand side of (3.9), we have

$$\left|\frac{2\alpha - e^{\iota\beta}G(z)}{e^{\iota\beta}G(z)}\right| = \left|\frac{2\alpha e^{-\iota\beta}f^{(p)}(z) - \left(f^{(p)}(z) + zf^{(p+1)}(z)\right)}{\left(f^{(p)}(z) + zf^{(p+1)}(z)\right)}\right|$$

$$\begin{aligned}\left|\frac{2\alpha - e^{\iota\beta}G(z)}{e^{\iota\beta}G(z)}\right| &= \left|\frac{2\alpha e^{-\iota\beta}f^{(p)}(z) - \left(f^{(p)}(z) + zf^{(p+1)}(z)\right)}{\left(f^{(p)}(z) + zf^{(p+1)}(z)\right)}\right| \\ &= \left|\frac{2\alpha e^{-\iota\beta} - 1 + \sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}(n+1)\,a_{n+p}2\alpha e^{-\iota\beta} - (n+1)}{1 + \sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}(n+1)^2\left|a_{n+p}\right|}\right| \\ &= \frac{\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}\left|(n+1)\,a_{n+p}\right|\left|(n+1) - 2\alpha e^{-\iota\beta}\right|}{1 - \sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}(n+1)^2\left|a_{n+p}\right|}.\end{aligned}$$

The last expression is bounded above by 1 if

$$\begin{aligned}&\left|1 - 2\alpha e^{-\iota\beta}\right| + \sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}\left|(n+1)\,a_{n+p}\right|\left|(n+1) - 2\alpha e^{-\iota\beta}\right| \\ \leq\ & 1 - \sum_{n=1}^{\infty}\frac{(p+n)!}{(n+1)!}(n+1)^2\left|a_{n+p}\right|\end{aligned} \tag{3.10}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < 1$. After simplification, inequality (3.10) can be written as

$$\sum_{n=1}^{\infty}\frac{(p+n)!}{n!}\left\{n + 1 + \left|(n+1) - 2\alpha e^{-\iota\beta}\right|\right\}\left|a_{n+p}\right| \leq 1 - \left|1 - 2\alpha e^{-\iota\beta}\right|.$$

This completes the proof. □

When we take $p = 1$, we have the following known result given in [13].

Corollary 3.10. *If $f \in A$ satisfies*

$$\sum_{n=2}^{\infty}\left\{n\left(n + \left|n - 2\alpha e^{-\iota\beta}\right|\right)\right\}|a_n| \leq 1 - \left|1 - 2\alpha e^{-\iota\beta}\right|$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f \in K_\beta(\alpha)$.

Taking $p = 1, \beta = 0$ in above theorem we have the following result given in [12].

Corollary 3.11. *Let $0 < \alpha < 1$. If $f \in A$ satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \le \frac{1}{2}(1-|1-2\alpha|) = \begin{cases} \alpha; & \left(0<\alpha\le\frac{1}{2}\right) \\ 1-\alpha; & \left(\frac{1}{2}<\alpha<1\right) \end{cases}$$

then $f(z) \in N(\alpha)$.

Taking $\beta = \frac{\pi}{4}$ in above Theorem we have the following result.

Corollary 3.12. *If* $f(z) \in A(p)$ *satisfies*

$$\sum_{n=1}^{\infty} \frac{(p+n)!}{n!}\left(n+1+\sqrt{(n+1)^2-2\sqrt{2}\alpha(n+1)+4\alpha^2}\right)|a_{n+p}|$$
$$\le \quad 1-\sqrt{1-2\sqrt{2}\alpha+4\alpha^2}$$

for some $0<\alpha<\frac{\sqrt{2}}{2}$, *then* $f(z) \in S_{\frac{\pi}{4}}(\alpha, p)$.

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Conflict of Interest

No potential conflict of interest was reported by the authors.

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Lp-analysis of one-dimensional repulsive Hamiltonian with a class of perturbations

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A B S T R A C T

The spectrum of one-dimensional repulsive Hamiltonian with a class of perturbations Hp = $-\frac{d^2}{dx^2} - x^2 + V(x)$ in $L^p(\mathbb{R})$ $(1 < p < \infty)$ is explicitly given. It is also proved that the domain of H_p is embedded into weighted L^q-spaces for some $q > p$. Additionally, non-existence of related Schrödinger $(C_0\text{-})$semigroup in $L^p(\mathbb{R})$ is shown when $V(x) \equiv 0$.

Keywords: repulsive Hamiltonian; WKB methods

1. Introduction

In this paper we consider

$$H := -\frac{d^2}{dx^2} - x^2 + V(x) \tag{1}$$

in $L^p(\mathbb{R})$, where $V \in C(\mathbb{R})$ is a real-valued and satisfies $V(x) \geq -a(1 + x^2)$ for some constant $a \geq 0$ and

$$\int_{\mathbb{R}} \frac{|V(x)|}{\sqrt{1 + x^2}}\, dx < \infty. \tag{2}$$

The operator (1) describes the quantum particle affected by a strong repulsive force from the origin. In fact, in the classical sense the corresponding Hamiltonian (functional) is given by $\hat{H}(x, p) = p^2 - x^2$ and then the particle satisfying $\dot{x} = \partial_p \hat{H}$ and $\dot{p} = -\partial_x \hat{H}$ goes away much faster than that for the free Hamiltonian $\hat{H}_0(x, p) = p^2$.

In the case where $p = 2$, the essential selfadjointness of H, endowed with the domain $C_0^\infty(\Omega)$,has been discussed by Ikebe and Kato [7]. After that several properties of H is found out in a mount of subsequent papers (for studies of scattering theory e.g., Bony et al. [2], Nicoleau [10] and also Ishida [8]).

In contrast, if p is different from 2, then the situation becomes complicated. Actually, papers which deals with the properties of H is quite few because of absence of good properties like symmetricity. In the L^p-

framework, it is quite useful to consider the accretivity and sectoriality of the second-order differential operators. In fact, the case $-\frac{d^2}{dx^2} + V(x)$ with a nonnegative potential V is formally sectorial in L^p, and therefore one can find many literature even N-dimensional case (e.g., Kato [9], Goldstein[6], Tanabe [14], Engel-Nagel [5]). However, it seems quite difficult to describe such a kind of non accretive operators in a certain unified theory in the literature.

The present paper is in a primary position to make a contribution for theory of non-accretive operators in L^p as mentioned above. The aim of this paper is to give a spectral properties of $H = -\frac{d^2}{dx^2} - x^2 + V(x)$ for the case where $V(x)$ can be regarded as a perturbation of the leading part $-\frac{d^2}{dx^2} - x^2$; note that if $V(x) = [\log(e + |x|)]^{-\alpha}$ ($\alpha \in \mathbb{R}$), then $\alpha < 1$ is admissible, which is same threshold as in the short range potential for $-\frac{d^2}{dx^2} - x^2$ stated in Bony [2] and also Ishida [8].

Here we define the minimal realization $H_{p,\min}$ of H in $L^p = L^p(\mathbb{R})$ as

$$\begin{cases} D(H_{p,\min}) := C_0^\infty(\mathbb{R}), \\ H_{p,\min}u(x) := -u''(x) - x^2u(x) + V(x)u(x). \end{cases} \tag{3}$$

Theorem 1.1. *For every $1 < p < \infty$, $H_{p,\min}$ is closable and the spectrum of the closure H_p is explicitly given as*

$$\sigma(H_p) = \left\{\lambda \in \mathbb{C} \ ; \ |\mathrm{Im}\,\lambda| \le \left|1 - \frac{2}{p}\right|\right\}.$$

Moreover, for every $1 < p < q < \infty$, one has consistence of the resolvent operators:

$$(\lambda + H_p)^{-1}f = (\lambda + H_q)^{-1}f \ a.e.\ on\ \mathbb{R} \quad \forall\, \lambda \in \rho(H_p) \cap \rho(H_q), \quad \forall\, f \in L^p \cap L^q.$$

Remark 1.1. If $p = 2$, then our assertion is nothing new. The crucial part is the case $p \neq 2$ which is the case where the symmetricity of H breaks down. The similar consideration for $-\frac{d^2}{dx^2} + V$ (but in L^2-setting) can be found in Dollard-Friedman [4].

This paper is organized follows: In Section 2, we prepare two preliminary results. In Section 3, we consider the fundamental systems of $\lambda u + Hu = 0$, and estimate the behavior of their solutions. By virtue of that estimates, we will describe the resolvent set of H_p in Section 4. In section 5, we prove never to be generated C_0-semigroups by $\pm iH_p$ under the condition $V = 0$.

2. Preliminary results

First we state well-known results for the essentially selfadjointness of Schrodinger operators in ¨ L^2which is firstly described in [7]. We would like to refer also Okazawa [12].

Theorem 2.1 (Okazawa [12, Corollary 6.11]). *Let $V(x)$ be locally in L^2 (R) and assume that $V(x) \ge -c^1 - c_2|x|^2$, where c1, c2 ≥ 0 are constants. Then $H_{2,min}$ is essentially selfadjoint.*

Next we note the asymptotic behavior of solutions to second-order linear ordinary differential equations of the form

$$y''(x) = (\Phi(x) + \Psi(x))y(x)$$

in which the term $\Psi(x)y(x)$ can be treated as a perturbation of the leading part $\Phi(x)y(x)$.

Theorem 2.2 (Olver [13, Theorem 6.2.2 (p.196)]). *In a given finite or infinite interval (a_1, a_2), let $a \in (a_1, a_2)$, $\Psi(x)$ a positive, real, twice continuously differentiable function, $\Psi(x)$ a continuous real or complex function, and*

$$F(x) = \int \left\{ \frac{1}{\Phi(x)^{1/4}} \frac{d^2}{dx^2} \left(\frac{1}{\Phi(x)^{1/4}} \right) - \frac{\Psi(x)}{\Phi(x)^{1/2}} \right\} dx.$$

Then in this interval the differential equation

$$\frac{d^2 w}{dx^2} = \{\Phi(x) + \Psi(x)\} w$$

has twice continuously differential solutions

such that

$$|\varepsilon_j(x)|, \ \frac{1}{\Phi(x)^{1/2}} |\varepsilon_j(x)| \le \exp\left\{ \frac{1}{2} \mathcal{V}_{a_j,x}(F) \right\} - 1 \quad (j = 1, 2)$$

provided that $\mathcal{V}_{a_j,x}(F) < \infty$ (where $\mathcal{V}_{a_j,x}(F) = \int |F'(t)|\, dt$ is the total variation of F). If $\Psi(x)$ is real, then the solutions $w_1(x)$ and $w_2(x)$ are complex conjugates.

For the above theorem, see also Beals-Wong [1, 10.12, p.355].

3. Fundamental systems of $\lambda u - u'' - x^2 u + Vu = 0$

3.1. The case $\lambda \in \mathbb{R}$

We consider the behavior of solutions to

$$\lambda u(x) - u''(x) - x^2 u(x) + V(x)u(x) = 0, \quad x \in \mathbb{R}, \tag{4}$$

where $\lambda \in \mathbb{R}$.

Proposition 3.1. *There exist solutions $u_{\lambda,1}, u_{\lambda,2}$ of (4) such that $u_{\lambda,1}$ and $u_{\lambda,2}$ are linearly independent and satisfy*

$$|u_{\lambda,1}(x)| \le C_\lambda (1 + |x|)^{-\frac{1}{2}}, \quad |u_{\lambda,2}(x)| \le C_\lambda (1 + |x|)^{-\frac{1}{2}} \quad \forall x \in \mathbb{R},$$

where $\lambda \in \mathbb{R}$.

Proposition 3.1. *There exist solutions $u_{\lambda,1}, u_{\lambda,2}$ of (4) such that $u_{\lambda,1}$ and $u_{\lambda,2}$ are linearly independent and satisfy*

$$|u_{\lambda,1}(x)| \le C_\lambda (1 + |x|)^{-\frac{1}{2}}, \quad |u_{\lambda,2}(x)| \le C_\lambda (1 + |x|)^{-\frac{1}{2}} \quad \forall x \in \mathbb{R},$$
$$|u_{\lambda,1}(x)| \ge \frac{1}{2} (1 + |x|)^{-\frac{1}{2}}, \quad |u_{\lambda,2}(x)| \ge \frac{1}{2} (1 + |x|)^{-\frac{1}{2}} \quad \forall x \ge R_\lambda$$

for some constants $C_\lambda, R_\lambda > 0$ independent of x. In particular, $u_{\lambda,1}$, $u_{\lambda,2} \in L^p(\mathbb{R})$ if and only if $2 < p < \infty$.

Proof. First we consider (4) for $x > 0$. Using the Liouville transform

$$v(y) := (2y)^{\frac{1}{4}} u\left((2y)^{\frac{1}{2}}\right), \qquad \text{or equivalently,} \qquad u(x) = x^{-\frac{1}{2}} v\left(\frac{x^2}{2}\right),$$

we have

$$(\lambda - x^2)x^{-\frac{1}{2}} v\left(\frac{x^2}{2}\right) = u''(x) - V(x)u(x) = x^{\frac{3}{2}} v''\left(\frac{x^2}{2}\right) + \frac{3}{4}x^{-\frac{5}{2}} v\left(\frac{x^2}{2}\right) - x^{-\frac{1}{2}} V(x) v\left(\frac{x^2}{2}\right).$$

Therefore noting that $y = x^2/2$, we see that

$$v''(y) = \left[-\left(1 - \frac{\lambda}{4y}\right)^2 + \frac{\lambda^2 - 3}{16y^2} + \frac{V((2y)^{\frac{1}{2}})}{2y}\right] v(y) = (\Phi(y) + \Psi(y))v(y). \tag{5}$$

Here we have put for $y > 0$,

$$\Phi(y) := -\left(1 - \frac{\lambda}{4y}\right)^2, \quad \Psi(y) := \frac{\lambda^2 - 3}{16y^2} + \frac{V((2y)^{\frac{1}{2}})}{2y}.$$

Let

$$\Pi(y) := |\Phi(y)|^{-\frac{1}{4}} \left(-\frac{d^2}{dx^2} + \Psi(y)\right) |\Phi(y)|^{-\frac{1}{4}}, \quad y \geq \lambda_+ := \max\{\lambda, 0\}.$$

Then we see that for every $y \geq \lambda_+$,

$$|\Pi(y)| \leq \left(1 - \frac{\lambda}{4y}\right)^{-3} \frac{3\lambda^2}{64y^2} + \left(1 - \frac{\lambda}{4y}\right)^{-2} \frac{\lambda}{4y^3} + \left(1 - \frac{\lambda}{4y}\right)^{-1} \frac{|\lambda^2 - 3|}{16y^2} + \frac{|V((2y)^{\frac{1}{2}})|}{2y} \leq \frac{M_\lambda}{y^2} + \frac{|V((2y)^{\frac{1}{2}})|}{2y},$$

where M_λ is a positive constant depending only on λ. Therefore

$$\int_{\lambda_+}^{\infty} |\Pi(y)|\, dy \leq M_\lambda \int_{\lambda_+}^{\infty} \frac{1}{y^2}\, dy + \int_{\sqrt{2\lambda_+}}^{\infty} \frac{|V(x)|}{x}\, dx < \infty.$$

Thus $\Pi \in L^1((\lambda_+, \infty))$. By Theorem 2.2, we obtain that there exists a fundamental system $(v_{\lambda,1}, v_{\lambda,2})$ of (5) such that

$$v_{\lambda,1}(y) y^{i\frac{\lambda}{4}} e^{-iy} \to 1, \quad v_{\lambda,2}(y) y^{-i\frac{\lambda}{4}} e^{iy} \to 1 \quad \text{as } y \to \infty$$

(see also [11]). Taking $u_{\lambda,j}(x) = x^{-\frac{1}{2}} v_{\lambda,j}(x^2/2)$ for $j = 1, 2$, we obtain that $(u_{\lambda,1}, u_{\lambda,2})$ is a fundamental system of (4) on (λ_+, ∞) and

$$u_{\lambda,1}(y) x^{\frac{1}{2} + i\frac{\lambda}{2}} e^{-i\frac{x^2}{2}} \to 2^{-i\frac{\lambda}{4}}, \quad u_{\lambda,2}(x) x^{\frac{1}{2} - i\frac{\lambda}{2}} e^{i\frac{x^2}{2}} \to 2^{i\frac{\lambda}{4}},$$

as $x \to \infty$. The above fact implies that there exists a constant $R_\lambda > \lambda_+$ such that

$$\frac{1}{2} x^{-\frac{1}{2}} \leq |u_{\lambda,j}(x)| \leq \frac{3}{2} x^{-\frac{1}{2}}, \quad x \geq R_\lambda, \quad j = 1, 2.$$

We can extend $(u_{\lambda,1}, u_{\lambda,2})$ as a fundamental system on $\mathbb{R}$. By applying the same argument as above to (4) for $x < 0$, we can construct a different fundamental system $(\tilde{u}_{\lambda,1}, \tilde{u}_{\lambda,2})$ on $\mathbb{R}$ satisfying

$$\frac{1}{2}|x|^{-\frac{1}{2}} \leq |\tilde{u}_{\lambda,j}(x)| \leq \frac{3}{2}|x|^{-\frac{1}{2}}, \quad x \leq -\tilde{R}_\lambda, \quad j = 1, 2.$$

By definition of fundamental system, $u_{\lambda,j}$ can be rewritten as

$$u_{\lambda,1}(x) = c_{11}\tilde{u}_{\lambda,1}(x) + c_{12}\tilde{u}_{\lambda,2}(x), \quad u_{\lambda,2}(x) = c_{21}\tilde{u}_{\lambda,1}(x) + c_{22}\tilde{u}_{\lambda,2}(x).$$

Hence we have the upper and lower estimates of $u_{\lambda,j}$ ($j = 1, 2$), respectively. □

3.2. *The case* $\lambda \in \mathbb{C} \setminus \mathbb{R}$

We consider the behavior of solutions to

$$\lambda u(x) - u''(x) - x^2 u(x) + V(x)u(x) = 0, \tag{6}$$

where $\lambda \in \mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Im}\lambda > 0$. The case $\operatorname{Im}\lambda < 0$ can be reduced to the problem $\operatorname{Im}\lambda > 0$ via complex conjugation.

3.2.1. Properties of solutions to an auxiliary problem

We start with the following function φ_λ:

$$\varphi_\lambda(x) := x^{-\frac{1+\lambda i}{2}} e^{i\frac{x^2}{2}}, \quad x > 0. \tag{7}$$

Then by a direct computation we have

Lemma 3.2. *φ_λ satisfies*

$$\lambda\varphi_\lambda - \varphi_\lambda'' - x^2\varphi_\lambda + g_\lambda\varphi_\lambda = 0, \quad x \in (0,\infty), \tag{8}$$

where $g_\lambda(x) := \frac{(1+\lambda i)(3+\lambda i)}{4x^2}$, $x > 0$.

Remark 3.1. If $\lambda = i$ or $\lambda = 3i$, then φ_λ is nothing but a solution of the original equation (6) with $V = 0$.

Next we construct another solution of (8) which is linearly independent of φ_λ. Before construction, we prepare the following lemma.

Lemma 3.3. *Let λ satisfy $\operatorname{Im}\lambda > 0$ and let φ_λ be given in* (7)*. Then for every $a > 0$, there exists $F_a^\lambda \in \mathbb{C}$ such that*

$$\int_a^x \varphi_\lambda(t)^{-2}\,dt \to F_a^\lambda \quad \text{as } x \to \infty$$

and then $x \mapsto \int_a^x \varphi_\lambda(t)^{-2}\,dt - F_a^\lambda$ is independent of a. Moreover, for every $x > 0$,

$$\left|\int_a^x \varphi_\lambda(t)^{-2}\,dt - F_a^\lambda - \frac{i}{2}x^{\lambda i}e^{-ix^2}\right| \le C_\lambda x^{-\operatorname{Im}\lambda - 2},$$

where $C_\lambda := \dfrac{|\lambda|}{4}\left(1 + \sqrt{1 + \left(\dfrac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda + 2}\right)^2}\right)$.

Remark 3.2. If $a = 0$ and $\lambda = i$, then F_0^i gives the Fresnel integral $\lim_{x\to\infty}\int_0^x e^{-it^2}\,dt$. Hence $F_0^i = \sqrt{\pi/8}(1-i)$.

Proof. By integration by part, we have

$$\int_a^x t^{1+\lambda i}e^{-it^2}\,dt = \left(\frac{i}{2}x^{\lambda i}e^{-ix^2} - \frac{i}{2}a^{\lambda i}e^{-ia^2}\right) + \frac{\lambda i}{4}\left(x^{\lambda i-2}e^{-ix^2} - a^{\lambda i-2}e^{-ia^2}\right) - \frac{\lambda i(\lambda i-2)}{4}\int_a^x t^{\lambda i-3}e^{-it^2}\,dt.$$

Noting that $t^{\lambda i-3}e^{-it^2}$ is integrable in (a,∞), we have

$$\int_a^x t^{1+\lambda i}e^{-it^2}\,dt \to -\frac{i}{2}a^{\lambda i}e^{-ia^2} - \frac{\lambda i}{4}a^{\lambda i-2}e^{-ia^2} - \frac{\lambda i(\lambda i-2)}{4}\int_a^\infty t^{\lambda i-3}e^{-it^2}\,dt =: F_a^\lambda$$

as $x \to \infty$. And therefore $\int_a^x t^{1+\lambda i} e^{-it^2}\, dt - F_a^\lambda$ is independent of a and

$$\left|\int_a^x t^{1+\lambda i} e^{-it^2}\, dt - F_a^\lambda - \frac{i}{2} x^{\lambda i} e^{-ix^2}\right| = \left|\frac{\lambda}{4} x^{-\lambda-2} e^{-ix^2} + \frac{\lambda i(\lambda i - 2)}{4} \int_x^\infty t^{\lambda i - 3} e^{-it^2}\, dt\right| \le C_\lambda x^{-\operatorname{Im}\lambda - 2}.$$

This is nothing but the desired inequality. □

Lemma 3.4. *Let φ_λ be as in (7) and define ψ_λ as*

$$\psi_\lambda(x) := \varphi_\lambda(x) \int_a^x \frac{1}{\varphi_\lambda(t)^2}\, dt - F_a^\lambda \varphi_\lambda(x), \quad x > 0. \tag{9}$$

Then ψ_λ is independent of a and $(\varphi_\lambda, \psi_\lambda)$ is a fundamental system of (8). Moreover, there exists $a_0 > 0$ such that

$$\frac{1}{3} x^{-\frac{\operatorname{Im}\lambda+1}{2}} \le |\psi_\lambda(x)| \le x^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x \in [a_0, \infty).$$

Proof. From Lemma 3.3 we have

$$x^{\frac{\operatorname{Im}\lambda+1}{2}} \left|\psi_\lambda(x) - \frac{i}{2} x^{-\frac{1-\lambda i}{2}} e^{-i\frac{x^2}{2}}\right| = x^{\frac{\operatorname{Im}\lambda+1}{2}} |\varphi_\lambda(x)| \left|\int_a^x \frac{1}{\varphi_\lambda(t)^2}\, dt - F_a^\lambda - \frac{i}{2} x^{\lambda i} e^{-ix^2}\right| \le C_\lambda x^{-2}.$$

Putting $a_0 = (6C_\lambda)^{\frac{1}{2}}$, we deduce the desired assertion. □

3.2.2. Fundamental system of the original problem

Next we consider

$$\lambda w - w'' - x^2 w + g_\lambda w = \tilde{g}_\lambda h, \quad x > 0 \tag{10}$$

with a given function h, where g_λ is given as in Lemma 3.2 and $\tilde{g}_\lambda := g_\lambda - V$. To construct solutions of (6), we will define two types of solution maps $h \mapsto w$ and consider their fixed points.

First we construct a solution of (6) which behaves like ψ_λ at infinity.

Definition 3.5. *For $b > 0$, define*

$$Uh(x) := \psi_\lambda(x) - \psi_\lambda(x) \int_b^x \varphi_\lambda(s) \tilde{g}_\lambda(s) h(s)\, ds - \varphi_\lambda(x) \int_x^\infty \psi_\lambda(s) \tilde{g}_\lambda(s) h(s)\, ds, \quad x \in [b, \infty)$$

for h belonging to a Banach space

$$X_\lambda(b) := \left\{ h \in C([b, \infty))\ ;\ \sup_{x \in [b,\infty)} \left(x^{\frac{\operatorname{Im}\lambda+1}{2}} |h(x)| \right) < \infty \right\}, \quad \|h\|_{X_\lambda(b)} := \sup_{x \in [b,\infty)} \left(x^{\frac{\operatorname{Im}\lambda+1}{2}} |h(x)| \right).$$

Remark 3.3. For arbitrary fixed $b > 0$, all solutions of (10) can be described as follows:

$$w_{c_1,c_2}(x) = c_1 \varphi_\lambda(x) + c_2 \psi_\lambda(x) + \int_b^x \Big(\varphi_\lambda(x) \psi_\lambda(s) - \varphi_\lambda(s) \psi_\lambda(x) \Big) \tilde{g}_\lambda(s) h(s)\, ds,$$

where $c_1, c_2 \in \mathbb{C}$. Suppose that $h \in C_0^\infty((b, \infty))$ with $\operatorname{supp} h \subset [b_1, b_2]$. Then $w_{c_1,c_2} \in C([b, \infty))$. In particular, for $x \ge b_2$,

$$w_{c_1,c_2}(x) = \left(c_1 + \int_{b_1}^{b_2} \psi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds\right)\varphi_\lambda(x) + \left(c_2 - \int_{b_1}^{b_2} \varphi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds\right)\psi_\lambda(x).$$

Therefore w_{c_1,c_2} behaves like ψ_λ (that is, $w_{c_1,c_2} \in X_\lambda(b)$) only when

$$c_1 = -\int_{b_1}^{b_2} \psi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds = -\int_{b}^{\infty} \psi_\lambda(s)\tilde{g}_\lambda(s)h(s)\,ds.$$

In Definition 3.5 we deal with such a solution with $c_2 = 1$.

Well-definedness of U in Definition 3.5 and its contractivity are proved in next lemma.

Lemma 3.6. *The following assertions hold*:

(i) *for every $b > 0$, the map $U : X_\lambda(b) \to X_\lambda(b)$ is well-defined*;

(ii) *there exists $b_\lambda > 0$ such that U is contractive in $X_\lambda(b_\lambda)$ with*

$$\|Uh_1 - Uh_2\|_{X_\lambda(b)} \le \frac{1}{5}\|h_1 - h_2\|_{X_\lambda(b)}, \quad h_1, h_2 \in X_\lambda(b_\lambda)$$

and then U has a unique fixed point $w_1 \in X_\lambda(b_\lambda)$;

(iii) *w_1 can be extended to a solution of* (6) *in $\mathbb{R}$ satisfying*

$$\frac{1}{12}x^{-\frac{\operatorname{Im}\lambda+1}{2}} \le |w_1(x)| \le 2x^{-\frac{\operatorname{Im}\lambda+1}{2}}, \quad x \in [b_\lambda, \infty).$$

Proof. **(i)** By Lemma 3.4 we have $\psi_\lambda \in X_\lambda(b)$. Therefore to prove well-definedness of U, it suffices to show that the second term in the definition of U belongs to $X_\lambda(b)$.

Let $h \in X_\lambda(b)$. Then for $x \in [b, \infty)$,

$$x^{\frac{\operatorname{Im}\lambda+1}{2}} \left|\varphi_\lambda(x)\int_x^\infty \psi_\lambda(s)\tilde{g}(s)h(s)\,ds\right| \le x^{\operatorname{Im}\lambda}\|h\|_X \int_x^\infty s^{-\operatorname{Im}\lambda-1}|\tilde{g}_\lambda(s)|\,ds \le \|h\|_X\|s^{-1}\tilde{g}_\lambda\|_{L^1(b,\infty)}$$

and

$$x^{\frac{\operatorname{Im}\lambda+1}{2}} \left|\psi_\lambda(x)\int_b^x \varphi_\lambda(s)\tilde{g}(s)h(s)\,ds\right| \le \|h\|_X \int_b^x s^{-1}|\tilde{g}_\lambda(s)|\,ds \le \|h\|_X\|s^{-1}\tilde{g}_\lambda\|_{L^1(b,\infty)}.$$

Hence we have $Uh \in C([b, \infty))$ and therefore $Uh \in X_\lambda(b)$, that is, $U : X_\lambda(b) \to X_\lambda(b)$ is well-defined.

(ii) Let $h_1, h_2 \in X_\lambda(b)$. Then we have

$$Uh_1(x) - Uh_2(x) = -\psi_\lambda(x)\int_b^x \varphi_\lambda(s)\tilde{g}_\lambda(s)(h_1(s) - h_2(s))\,ds - \varphi_\lambda(x)\int_x^\infty \psi_\lambda(s)\tilde{g}_\lambda(s)(h_1(s) - h_2(s))\,ds.$$

Proceeding the same computation as above, we deduce

$$\|Uh_1 - Uh_2\|_{X_\lambda(b)} \le 2\|s^{-1}\tilde{g}_\lambda\|_{L^1(b,\infty)}\|h_1 - h_2\|_{X_\lambda(b)}.$$

Choosing b large enough, we obtain $\|Uh_1 - Uh_2\|_{X_\lambda(b)} \le 5^{-1}\|h_1 - h_2\|_{X_\lambda(b)}$, that is U is contractive in $X_\lambda(b)$. By contraction mapping principle, we obtain that U has a unique fixed point $w_1 \in X_\lambda(b)$.

(iii) Since w_1 satisfies (10) with $h = w_1$, w_1 is a solution of the original equation (6) in $[b, \infty)$. As in the last part of the proof of Proposition 3.1, we can extend w_1 as a solution of (6) in $\mathbb{R}$. Since $Uw_1 = w_1$ and $U0 = \psi_\lambda$, it follows from the contractivity of U that

$$\|w_1 - \psi_\lambda\|_X = \|Uw_1 - U0\|_X \le \frac{1}{5}\|w_1\|_X \le \frac{1}{5}\|w_1 - \psi_\lambda\|_X + \frac{1}{5}\|\psi_\lambda\|_X.$$

Consequently, we have $\|w_1 - \psi_\lambda\|_X \le 4^{-1}\|\psi_\lambda\|_X \le 4^{-1}$ and then for $x \ge b$,

$$|w_1(x)| \ge |\psi_\lambda(x)| - |w_1(x) - \psi_\lambda(x)| \ge \left(\frac{1}{3} - \|w_1 - \psi_\lambda\|_X\right) x^{-\frac{\mathrm{Im}\lambda+1}{2}} \ge \frac{1}{12} x^{-\frac{\mathrm{Im}\lambda+1}{2}}.$$

□

Next we construct another solution of (6) which behaves like φ_λ at infinity.

Definition 3.7. *Let $b > 0$ be large enough. Define*

$$\widetilde{U}h(x) := \varphi_\lambda(x) + \int_b^x \Big(\varphi_\lambda(x)\psi_\lambda(s) - \varphi_\lambda(s)\psi_\lambda(x)\Big)\tilde{g}_\lambda(s)h(s)\, ds$$

for h belonging to a Banach space

$$Y_\lambda(b) := \left\{h \in C([b,\infty))\ ;\ \sup_{x\in[b,\infty)} \left(x^{-\frac{\mathrm{Im}\lambda-1}{2}}|h(x)|\right) < \infty\right\},\quad \|h\|_{Y_\lambda(b)} := \sup_{x\in[b,\infty)} \left(x^{-\frac{\mathrm{Im}\lambda-1}{2}}|h(x)|\right).$$

Lemma 3.8. *The following assertions hold*:

(i) *for every $b > 0$, the map $\widetilde{U} : Y_\lambda(b) \to Y_\lambda(b)$ is well-defined*;

(ii) *there exists $b_\lambda > 0$ such that $\widetilde{U}$ is contractive in $Y_\lambda(b_\lambda)$ with*

$$\|\widetilde{U}h_1 - \widetilde{U}h_2\|_{Y_\lambda(b)} \le \frac{1}{5}\|h_1 - h_2\|_{Y_\lambda(b)}, \quad h_1, h_2 \in Y_\lambda(b_\lambda)$$

and then $\widetilde{U}$ has a unique fixed point $\tilde{w}_1 \in Y_\lambda(b_\lambda)$;

(iii) *$\tilde{w}_1$ can be extended to a solution of* (6) *in $\mathbb{R}$ satisfying*

$$\frac{1}{2}x^{\frac{\mathrm{Im}\,\lambda-1}{2}} \le |\tilde{w}_1(x)| \le 2x^{\frac{\mathrm{Im}\,\lambda-1}{2}}, \quad x \in [b_\lambda, \infty).$$

Proof. The proof is similar to the one of Lemma 3.6. □

Considering the equation (6) for $x < 0$, we also obtain the following lemma.

Lemma 3.9. *For every $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda > 0$, there exist a fundamental system (w_1, w_2) of* (6) *and positive constants $c_\lambda, C_\lambda, R_\lambda$ such that*

$$|w_1(x)| \le C_\lambda(1+|x|)^{\frac{\mathrm{Im}\,\lambda-1}{2}}, \quad x \le 0, \qquad |w_1(x)| \le C_\lambda(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}, \quad x \ge 0, \tag{11}$$

$$|w_2(x)| \le C_\lambda(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}, \quad x \le 0, \qquad |w_2(x)| \le C_\lambda(1+|x|)^{\frac{\mathrm{Im}\,\lambda-1}{2}}, \quad x \ge 0 \tag{12}$$

and

$$|w_1(x)| \ge c_\lambda(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}, \quad x \ge R_\lambda, \quad |w_2(x)| \ge c_\lambda(1+|x|)^{-\frac{\mathrm{Im}\,\lambda+1}{2}}, \quad x \le -R_\lambda. \tag{13}$$

Proof. In view of Lemma 3.6, it suffices to find w_2 satisfying the conditions above.

Let w_* and $\tilde{w}_*$ be given as in Lemmas 3.6 and 3.8 with $V(x)$ replaced with $V(-x)$. Noting that w_1 can be rewritten as $w_1(x) = c_1 w_*(-x) + c_2\tilde{w}_*(-x)$, we see from Lemma 3.6 and 3.8 that (11) and the first half of (13) are satisfied. Set $w_2(x) = w_*(-x)$ for $x \in \mathbb{R}$. As in the same way, we can verify (12).

Finally, we prove the last half of (13). Since $H_{2,\min}$ is essentially selfadjoint in $L^2(\mathbb{R})$, λ belongs to the resolvent set of H_2, that is, $N(\lambda + H_2) = \{0\}$. This implies that $w_2 \notin L^2(\mathbb{R})$. Noting that $w_2 \in L^2((-\infty, 0))$, we have $w_2 \notin L^2((0,\infty))$. Now using the representation

$$w_2(x) = c_1 w_1(x) + c_2 \tilde{w}_1(x), \quad x \in \mathbb{R},$$

we deduce that $c_2 \neq 0$. Therefore using Lemma 3.6 **(iii)** and Lemma 3.8 **(iii)**, we have

$$|w_2(x)| \geq |c_2|\,|\tilde{w}_1(x)| - |c_1|\,|w_1(x)| \geq \frac{|c_2|}{2} x^{\frac{\operatorname{Im}\lambda - 1}{2}} - 2|c_1|\, x^{-\frac{\operatorname{Im}\lambda+1}{2}} \geq \frac{|c_2|}{4} x^{\frac{\operatorname{Im}\lambda-1}{2}}$$

for x large enough. □

4. Resolvent estimates in L^p

The following lemma, verified by the variation of parameters, gives a possibility of representation of the Green function for resolvent operator H in L^p.

Lemma 4.1. *Assume that $\lambda \in \rho(\widetilde{H})$ in L^p, where $\widetilde{H}$ is a realization of H in L^p. Then for every $u \in C_0^\infty(\mathbb{R})$,*

$$u(x) = \frac{w_1(x)}{W_\lambda} \int_{-\infty}^{x} w_2(s) f(s)\, ds + \frac{w_2(x)}{W_\lambda} \int_{x}^{\infty} w_1(s) f(s)\, ds, \quad x \in \mathbb{R},$$

where $f := \lambda u - u'' - x^2 u + Vu \in C_0^\infty(\mathbb{R})$ and $W_\lambda \neq 0$ is the Wronskian of (w_1, w_2).

Proposition 4.2. *Let $1 < p < \infty$. If $|1 - \frac{2}{p}| < \operatorname{Im}\lambda$, then the operator defined as*

$$R(\lambda) f(x) := \frac{w_1(x)}{W_\lambda} \int_{-\infty}^{x} w_2(s) f(s)\, ds + \frac{w_2(x)}{W_\lambda} \int_{x}^{\infty} w_1(s) f(s)\, ds, \quad f \in C_0^\infty(\mathbb{R})$$

can be extended to a bounded operator on L^p. More precisely, there exists $M_\lambda > 0$ such that

$$\|R(\lambda) f\|_{L^p} \leq M_\lambda \left[|\operatorname{Im}\lambda|^2 - \left(1 - \frac{2}{p}\right)^2 \right]^{-1} \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}). \tag{14}$$

In particular, $H_{p,\min}$ is closable and its closure H_p satisfies

$$\left\{ \lambda \in \mathbb{C}\, ;\ |\operatorname{Im}\lambda| > \left|1 - \frac{2}{p}\right| \right\} \subset \rho(H_p).$$

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Set

$$u_1(x) := w_1(x) \int_{-\infty}^{x} w_2(s) f(s)\, ds, \quad u_2(x) := w_1(x) \int_{x}^{\infty} w_1(s) f(s)\, ds.$$

We divide the proof of $u_1 \in L^p(\mathbb{R})$ into two cases $x \geq 0$ and $x < 0$; since the proof of $u_2 \in L^p(\mathbb{R})$ is similar, this part is omitted.

The case u_1 for $x \geq 0$, it follows from Lemma 3.9 and Hölder inequality that

$$\begin{aligned} |u_1(x)| &\leq C_\lambda^2 (1 + |x|)^{-\frac{\operatorname{Im}\lambda+1}{2}} \left[\int_{-\infty}^{0} (1 + |s|)^{-\frac{\operatorname{Im}\lambda+1}{2}} |f(s)|\, ds + \int_0^x (1 + |s|)^{\frac{\operatorname{Im}\lambda - 1}{2}} |f(s)|\, ds \right] \\ &\leq C_\lambda^2 \left(\frac{\operatorname{Im}\lambda + 1}{2} p' - 1 \right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_-)} (1 + |x|)^{-\frac{\operatorname{Im}\lambda+1}{2}} \end{aligned}$$

$$
\begin{aligned}
&+ C_\lambda^2(1+|x|)^{-\frac{\operatorname{Im}\lambda+1}{2}} \left(\int_0^x (1+|s|)^{\frac{\operatorname{Im}\lambda-1}{2}p'-\alpha p'}\,ds\right)^{\frac{1}{p'}} \left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\le C_\lambda^2\left(\frac{\operatorname{Im}\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_-)}(1+|x|)^{-\frac{\operatorname{Im}\lambda+1}{2}} \\
&+ C_\lambda^2\left(\frac{\operatorname{Im}\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{1}{p'}} (1+|x|)^{-\frac{1}{p}-\alpha}\left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)^{\frac{1}{p}}
\end{aligned} \tag{15}
$$

with $0 < \alpha < \frac{\operatorname{Im}\lambda+1}{2} + 1/p'$. By the triangle inequality we have

$$
\|u_1\|_{L^p(\mathbb{R}_+)} \le C_\lambda^2\left(\frac{\operatorname{Im}\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\left(\frac{\operatorname{Im}\lambda+1}{2}p-1\right)^{-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}_-)} + \mathcal{I}_1(\alpha)
$$

and

$$
\begin{aligned}
(\mathcal{I}_1(\alpha))^p &= C_\lambda^{2p}\left(\frac{\operatorname{Im}\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{p}{p'}} \int_0^\infty (1+|x|)^{-1-\alpha p}\left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)dx \\
&= C_\lambda^{2p}\left(\frac{\operatorname{Im}\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{p}{p'}} (\alpha p)^{-1}\int_0^\infty |f(s)|^p\,ds.
\end{aligned}
$$

Choosing $\alpha = \frac{1}{pp'}\left(\frac{\operatorname{Im}\lambda-1}{2}p'+1\right)$, we obtain

$$
\|u_1\|_{L^p(\mathbb{R}_+)} \le C_\lambda^2\left(\frac{\operatorname{Im}\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\left(\frac{\operatorname{Im}\lambda+1}{2}p-1\right)^{-\frac{1}{p}}\|f\|_{L^p(\mathbb{R}_-)} + C_\lambda^2\left(\frac{\operatorname{Im}\lambda-1}{2}+\frac{1}{p'}\right)^{-1}\|f\|_{L^p(\mathbb{R}_+)}.
$$

The case u_1 for $x<0$, by the same way as the case $x>0$, we have

$$
|u_1(x)|^p \le C_\lambda^{2p}\left(\frac{\operatorname{Im}\lambda+1}{2}p'-\beta p'-1\right)^{-\frac{p}{p'}}(1+|x|)^{-1+\beta p}\int_{-\infty}^x (1+|s|)^{-\beta p}|f(s)|^p\,ds, \tag{16}
$$

where $0<\beta<\frac{\operatorname{Im}\lambda+1}{2}-\frac{1}{p'}$. Taking $\beta = \frac{1}{pp'}\left(\frac{\operatorname{Im}\lambda+1}{2}p'-1\right)$, we have

$$
\|u_1\|_{L^p(\mathbb{R}_-)} \le C_\lambda^2\left(\frac{\operatorname{Im}\lambda+1}{2}-\frac{1}{p'}\right)^{-1}\|f\|_{L^p(\mathbb{R}_-)}.
$$

Proceeding the same argument for u_2 and combining the estimates for u_1 and u_2, we obtain (14). □

Corollary 4.3. *Let $\mathcal{R}(\lambda)$ be as in Proposition 4.2. Then for every $f \in L^p(\mathbb{R})$, $\mathcal{R}(\lambda)f \in C(\mathbb{R})$ and*

$$
\sup_{x\in\mathbb{R}}\left((1+|x|)^{\frac{1}{p}}|\mathcal{R}(\lambda)f(x)|\right) \le \tilde{C}_\lambda\|f\|_{L^p}. \tag{17}
$$

Proof. Let $f \in C_0^\infty(\mathbb{R})$ and set u_1 and u_2 as in the proof of Proposition 4.2. Since the proof for u_1 and u_2 are similar, we only show the estimate of u_1. From (15), we have for $x \ge 0$,

$$
\begin{aligned}
(1+|x|)^{\frac{1}{p}}|u_1(x)| &\le C_\lambda^2\left(\frac{\operatorname{Im}\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}_-)}(1+|x|)^{-\frac{\operatorname{Im}\lambda}{2}+\frac{1}{p}-\frac{1}{2}} \\
&\quad + C_\lambda^2\left(\frac{\operatorname{Im}\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{1}{p'}}(1+|x|)^{-\alpha}\left(\int_0^x (1+|s|)^{\alpha p}|f(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\le C_\lambda^2\left(\frac{\operatorname{Im}\lambda+1}{2}p'-1\right)^{-\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}_-)} + C_\lambda^2\left(\frac{\operatorname{Im}\lambda-1}{2}p'-\alpha p'+1\right)^{-\frac{1}{p'}}\|f\|_{L^p(\mathbb{R}_+)},
\end{aligned}
$$

where $0 < \alpha < \frac{\mathrm{Im}\lambda+1}{2} + \frac{1}{p'}$. This implies (17) for $x \geq 0$. If $x \leq 0$, then from (16) we can obtain

$$(1+|x|)^{\frac{1}{p}}|u_1(x)| \leq C_\lambda^2 \left(\frac{\mathrm{Im}\lambda + 1}{2} p' - \beta p' - 1 \right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_-)},$$

where $0 < \beta < \frac{\mathrm{Im}\lambda+1}{2} - \frac{1}{p'}$. This yields (17) for $x \leq 0$. The proof is completed. □

By interpolation inequality, we deduce the following assertion.

Proposition 4.4. *Let $1 < p < \infty$ and $p \leq q \leq \infty$. Then*

$$D(H_p) \subset \left\{ w \in C(\mathbb{R})\ ;\ \langle x \rangle^{\frac{1}{p}-\frac{1}{q}} w \in L^q \right\}.$$

More precisely, there exists a constant $C_{p,q} > 0$ such that

$$\left\| \langle x \rangle^{\frac{1}{p}-\frac{1}{q}} u \right\|_{L^q} \leq C_{p,q} \left(\|H_p u\|_{L^p} + \|u\|_{L^p} \right), \quad u \in D(H_p).$$

Proof. The assertion follows from Proposition 4.2 and Corollary 4.3. □

Proposition 4.5. **(i)** *If $2 < p < \infty$ and $0 < |\mathrm{Im}\,\lambda| < 1 - \frac{2}{p}$, then $N(\lambda + H_p) \neq \{0\}$, and then*

$$\left\{ \lambda \in \mathbb{C}\ ;\ |\mathrm{Im}\,\lambda| \leq 1 - \frac{2}{p} \right\} \subset \sigma(H_p);$$

(ii) *If $1 < p < 2$ and $0 < |\mathrm{Im}\,\lambda| < \frac{2}{p} - 1$, then $\overline{N(\lambda + H_p)} \subsetneq L^p$, and then*

$$\left\{ \lambda \in \mathbb{C}\ ;\ |\mathrm{Im}\,\lambda| \leq \frac{2}{p} - 1 \right\} \subset \sigma(H_p).$$

$$\int_{-\infty}^{\infty} (\lambda u + H_p u) w_1 \, dx = \int_{-\infty}^{\infty} u(\lambda w_1 + H_{p'} w_1) \, dx = 0,$$

the closure of $R(\lambda + H_p)$ does not coincide with L^p, that is, $\overline{R(\lambda + H_p)} \subsetneq L^p$.

Since $\sigma(H_p)$ is closed in $\mathbb{C}$ and we can argue the same assertion for $\mathrm{Im}\lambda < 0$ via complex conjugation, we obtain the assertion. □

Combining the assertions above, we finally obtain Theorem 1.1.

5. Absence of C_0-semigroups on L^p ($p \neq 2$, $V = 0$)

In Theorem 1.1, we do not prove any assertions related to generation of C_0-semigroups by $\pm iH_p$. In this subsection we prove

Theorem 5.1. *Neither iH_p nor $-iH_p$ generates C_0-semigroup on L^p.*

Proof. We argue by a contradiction. Assume that iH_p generates a C_0-semigroup $T(t)$ on L^p. Then it follows from Theorem 1.1 (the coincidence of resolvent operators) that we have $T(t)f = S(t)f$ for every $t > 0$ and $f \in L^2 \cap L^p$, where $S(t)$ is the C_0-group generated by the skew-adjoint operator iH_2.

Fix $f_0 \in L^2 \cap L^p$ such that $\mathcal{F} f_0 \notin L^p$ ($\mathcal{F}$ is the Fourier transform). Then by the Mehler's formula (see e.g., Cazenave [3, Remark 9.2.5]), we see that

$$[S(t)]f(x) = \left(\frac{1}{2\pi\sinh(2t)}\right)^{\frac{N}{2}} e^{-i\frac{1}{2\tanh(2t)}|x|^2} \int_{-\infty}^{\infty} e^{-\frac{i}{\sinh(2t)}x\cdot y} e^{-i\frac{1}{2\tanh(2t)}|y|^2} f(y)\, dy.$$

In other words, using the operators

$$M_\tau g(x) := e^{-i\frac{|x|^2}{2\tau}} g(x), \quad D_\tau g(x) := \tau^{-\frac{N}{2}} g(\tau^{-1}x),$$

we can rewrite $S(t)$ as the following form $S(t)f = M_{\tanh(2t)}\mathcal{F} D_{\sinh(2t)} M_{\tanh(2t)} f$. Taking $f_{t_0} = M^{-1}_{\tanh(2t_0)} D^{-1}_{\sinh(2t_0)} f_0 \in L^p$, we have

$$S(t_0) f_{t_0} = M_{\tanh(2t)} \mathcal{F} f_0 \notin L^p.$$

This contradicts the fact $T(t_0) f_{t_0} \in L^p$. This completes the proof. □

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Conflict of Interest

All authors declare no conflicts of interest in this paper.

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Asymptotic stability of degenerate stationary solution to a system of viscous conservation laws in half line

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ABSTRACT

In this paper, we study a system of viscous conservation laws given by a form of a symmetric parabolic system. We consider the system in the one-dimensional half space and show existence of a degenerate stationary solution which exists in the case that one characteristic speed is equal to zero.Then we show the uniform a priori estimate of the perturbation which gives the asymptotic stability ofthe degenerate stationary solution. The main aim of the present paper is to show the a priori estimatewithout assuming the negativity of non-zero characteristics. The key to proof is to utilize the Hardyinequality in the estimate of low order terms.

Keywords: stationary waves; boundary layer solutions; compressible viscous gases; energy method; center manifold theory

1. Introduction

We consider a large-time behavior of solutions to a system of viscous conservation laws

$$u_t + f(u)_x = (Bu_x)_x \tag{1.1}$$

over a one-dimensional half line $\mathbb{R}_+ := (0, \infty)$. Here m is a positive integer; $u = u(t, x) \in \mathbb{R}^m$ is an unknown m-vector function; $f(u) \in \mathbb{R}^m$ is a flux function which is a smooth given function of u; B is a viscosity matrix which is an $m \times m$ symmetric and positive constant matrix.

We prescribe an initial condition for (1.1) as

$$u(0, x) = u_0(x) \quad (x \in \mathbb{R}_+), \tag{1.2}$$

where $u_0(x)$ is an initial data satisfying $u_0(x) \to 0$ as $x \to \infty$. We also put a Dirichlet boundary condition

$$u(t, 0) = u_b \quad (t > 0), \tag{1.3}$$

where $u_b \in \mathbb{R}^m$ is a constant.

Related to the system (1.1), existence and asymptotic stability of a boundary layer solution, which is a smooth stationary solution connecting a boundary data and a spatial asymptotic data, for modelsystems of compressible viscous gases are proved in the papers [1, 4, 8, 9, 10]. These results are generalized in the papers [7, 14] for a quasi-linear symmetric system of hyperbolic equations and parabolic equations under the stability condition discussed in [3, 11, 13]. Especially, in order to prove asymptotic stability of a degenerate boundary layer solution, which exists if the corresponding inviscid system has a characteristic field with speed zero, it is assumed in [7] that the speed of the non-zero characteristicsare negative. In the paper [6], the simplified system (1.1) with the flux function $f(u)$ given by the following form and satisfying the following assumption [A1] of symmetricity is considered:

$$f(u) = Au + \frac{1}{2}F(u,u), \tag{1.4}$$

where $A = (a_1, \dots, a_m)$ ($a_j \in \mathbb{R}^m$) is a constant $m \times m$ matrix; $F(\cdot,\cdot)$ is a bilinear map on $\mathbb{R}^m$ of the form

$$F(u,v) = \sum_{i,j=1}^{m} f_{ij} u_j v_i = \begin{pmatrix} \langle F_1 u, v\rangle \\ \vdots \\ \langle F_m u, v\rangle \end{pmatrix} \in \mathbb{R}^m$$

for $u = {}^t(u_1, \dots, u_m), v = {}^t(v_1, \dots, v_m) \in \mathbb{R}^m$, where $f_{ij} = {}^t(f_{ij}^1, \dots, f_{ij}^m) \in \mathbb{R}^m$ $(i, j = 1, \dots, m)$ are constant vectors and $F_k = (f_{ij}^k)_{ij}$ $(k = 1, \dots, m)$ are constant $m \times m$ matrices.

Assumption [A1]. (i) The matrix A is symmetric.

(ii) The bilinear map $F(\cdot,\cdot)$ is symmetric in the sense of $f_{ij}^k = f_{ji}^k = f_{kj}^i$.

From Assumption [A1], we see that $f_{ij} = f_{ji}$ and $F(u,v) = F(v,u)$, so that F_k is symmetric.

In the paper [6], the simplified system (1.1) with non-positive characteristics is considered and the convergence rate of solutions toward the degenerate boundary layer solution is obtained provided thatthe initial perturbation belongs to the weighted L^2space. The important property of the system in [6]is a negativity of non-zero characteristics which enable us to obtain the weighted L^2estimate.

The aim of the present paper is to show asymptotic stability of the degenerate boundary layer solution for (1.1) without assuming that the initial perturbation belongs to weighted spaces. Namely we show the uniform a priori estimate (3.4) under the assumption [A4]-(i) which means that the characteristic speed of the system is non-positive. The key to proof is to utilize a weight function defined in(3.9). In the case if the viscosity effect is strong enough, we can also show the estimate (3.4) withoutassuming the negativity of non-zero characteristics. This case corresponds to the assumption [A4]-(ii).To study this problem, we prescribe the following assumption.

Assumption [A2]. The matrix A has a simple zero-eigenvalue.

Note that Assumption [A2] corresponds to analysis on the transonic flow for the model system of compressible viscous gases studied in the papers [1, 2, 9, 12].

Notations. For vectors $u, v \in \mathbb{R}^m$, $|u|$ denotes the Euclidean norm of u; $\langle u, v\rangle$ denotes the Euclidean inner product of u and v. For real matrices A and B of which eigenvalues are real number, we use a notation $A \sim B$ if the numbers of positive eigenvalues, negative eigenvalues and zero eigenvalues of A coincide with those of B. For $p \in [1, \infty]$, L^p denotes a standard Lebesgue space over $\mathbb{R}_+$ equipped with a norm $\|\cdot\|_{L^p}$.

2. Existence of stationary solution

In this section, we summarized the existence result of the degenerate boundary layer solution studied in [6, 7]. Let $\tilde{u} = \tilde{u}(x)$ be a boundary layer solution, which is a smooth solution to a system of equations

$$f(\tilde{u})_x = (B\tilde{u}_x)_x \quad (x \in \mathbb{R}_+), \tag{2.1}$$

which is rewritten to

$$\tilde{u}_x = B^{-1}A\tilde{u} + \frac{1}{2}B^{-1}F(\tilde{u}, \tilde{u}) \tag{2.2}$$

by integrating (2.1) over (x, ∞) with using $\tilde{u}_x(x) \to 0$ as $x \to \infty$. We prescribe boundary conditions for $\tilde{u}$ as

$$\tilde{u}(0) = u_{\mathrm{b}}, \tag{2.3}$$

$$\tilde{u}(x) \to 0 \quad (x \to \infty). \tag{2.4}$$

To solve the above stationary problem, we introduce a following lemma proved in [6, 7].

Lemma 2.1 ([6, 7]). *Let B be a symmetric and positive definite matrix and A be a symmetric matrix.*

(i) *The matrix $B^{-1}A$ is diagonalizable.*

(ii) *There exists an orthogonal matrix Q such that $P := B^{-1/2}Q$ diagonalizes the matrix $B^{-1}A$ and satisfies ${}^tP = P^{-1}B^{-1}$.*

(iii) *$B^{-1}A \sim A$.*

Here $B^{1/2}$ is a symmetric and positive definite matrix satisfying $(B^{1/2})^2 = B$ and $B^{-1/2}$ is an inverse matrix of $B^{1/2}$.

We give a brief outline of proof of the solvability theorem to the stationary problem (2.2)–(2.4) by following the argument in [6, 7]. Due to Lemma 2.1, there exists a matrix P of the form

$$P = (r, P_*), \quad P_* : \ m \times (m-1) \text{ matrix},$$

which diagonalizes $B^{-1}A$, that is,

$$P^{-1}B^{-1}AP = \begin{pmatrix} 0 & {}^t\mathbf{0} \\ \mathbf{0} & \Lambda \end{pmatrix}, \tag{2.5}$$

where $\mathbf{0}$ is a column zero-vector and Λ is a diagonal $(m-1)\times(m-1)$ matrix satisfying $\det\Lambda \neq 0$. Note that the column vector r is an eigenvector of $B^{-1}A$ corresponding to the zero-eigenvalue, that is, $B^{-1}Ar = 0$ and hence $Ar = 0$, which yields that r is also an eigenvector of A corresponding to the zero-eigenvalue. We employ a new unknown function $\tilde{w}(x) := P^{-1}\tilde{u}(x)$ and deduce the system (2.2) to that for $\tilde{w}$ as

$$\tilde{w}_{1x} = g_1(\tilde{w}), \tag{2.6a}$$

$$\tilde{w}_{*x} = \Lambda \tilde{w}_* + g_*(\tilde{w}), \tag{2.6b}$$

where

$$\tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_* \end{pmatrix}, \quad \tilde{w}_1 = \tilde{w}_1(x) \in \mathbb{R}, \quad \tilde{w}_* = \tilde{w}_*(x) \in \mathbb{R}^{m-1},$$

$$g(\tilde{w}) = \begin{pmatrix} g_1(\tilde{w}) \\ g_*(\tilde{w}) \end{pmatrix} := \frac{1}{2} P^{-1} B^{-1} F(P\tilde{w}, P\tilde{w}), \quad g_1(\tilde{w}) \in \mathbb{R}, \quad g_*(\tilde{w}) \in \mathbb{R}^{m-1}.$$

Let $z = z(x) \in \mathbb{R}$ be a solution to (2.6a) restricted on the local center manifold $\tilde{w}_* = \Phi^c(\tilde{w}_1)$. Namely, $z(x)$ satisfies

$$z_x = g_1(z, \Phi^c(z)) = \kappa z^2 + O(|z|^3), \tag{2.7}$$

where κ is a constant given by

$$\kappa := \frac{1}{2} \langle r, F(r, r) \rangle. \tag{2.8}$$

Here the second equality in (2.7) is obtained by using $P^{-1}B^{-1} = {}^tP$ and $P\tilde{w} = r\tilde{w}_1 + P_*\tilde{w}_*$ as well as a bilinearity of F. To solve the equation (2.7), we put the following assumption.

Assumption [A3]. Let r be an eigenvector of the matrix A corresponding to the zero-eigenvalue. Then it is assumed that the constant κ defined by (2.8) is not equal to zero.

Note that we assume $\kappa < 0$ without loss of generality. We also note that the assumption [A3] is equivalent to the genuine nonlinearity of the zero-characteristic field.

If the boundary data u_b belongs to a certain region $\mathscr{M} \subset \mathbb{R}^m$, which is a one side of a neighborhood of the equilibrium divided by a local stable manifold, and if the boundary strength $\delta = |u_b|$ is sufficiently small, then the equation (2.7) has a solution z satisfying

$$\begin{aligned} &z(x) > 0, \quad z_x(x) < 0, \quad z(x) \sim \frac{\delta}{1 + \delta x}, \\ &|\partial_x^k z(x)| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} \quad (k = 0, 1, \ldots). \end{aligned} \tag{2.9}$$

By virtue of the center manifold theory, the solution $\tilde{w}$ to (2.6) is given by using z as

$$\tilde{w}_1(x) = z(x) + O(\delta e^{-cx}),$$

$$\tilde{w}_*(x) = \Phi^c(z(x)) + O(\delta e^{-cx}),$$

which gives the existence of the solution $\tilde{u}$.

Theorem 2.2 ([6, 7]). *Assume that Assumptions* [A1], [A2] *and* [A3] *hold. Then there exists a certain region* $\mathscr{M} \subset \mathbb{R}^m$ *such that if* $u_b \in \mathscr{M}$ *holds and* $\delta = |u_b|$ *is sufficiently small, then the problem* (2.2)–(2.4) *has a unique smooth solution* $\tilde{u}(x)$ *satisfying*

$$\tilde{u}(x) = rz(x) + O(z(x)^2 + \delta e^{-cx}), \tag{2.10}$$

$$\tilde{u}_x(x) = \kappa r z(x)^2 + O(z(x)^3 + \delta e^{-cx}). \tag{2.11}$$

3. Energy estimates

In this section, we show the uniform a priori estimate of a perturbation

$$\varphi(t,x) := u(t,x) - \tilde{u}(x)$$

which gives the existence of a solution globally in time. From (1.1) and (2.1), the equation for φ is given by

$$\varphi_t + D_u f(u)\varphi_x = B\varphi_{xx} - (D_u f(u) - D_u f(\tilde{u}))\tilde{u}_x \quad (x \in \mathbb{R}_+,\ t > 0), \tag{3.1}$$

where $D_u f(u) = A + (\langle f_{ij}, u\rangle)_{ij}$. The initial condition and the boundary condition are prescribed as

$$\varphi(0,x) = \varphi_0(x) := u_0(x) - \tilde{u}(x) \quad (x \in \mathbb{R}_+), \tag{3.2}$$

$$\varphi(t,0) = 0 \quad (t > 0). \tag{3.3}$$

To obtain the uniform a priori estimate, we prescribe the following assumption.

Assumption [A4]. It is assumed that either of the following two conditions is satisfied:

(i) The matrix A is non-positive definite, that is, the diagonal matrix Λ is negative definite, or

(ii) The viscosity effect is strong enough to satisfy

$$\langle B\phi, \phi\rangle > C_1 \tilde{F}|\phi|^2, \quad \tilde{F} := \max_{i,j,k} |f_{ij}^k|$$

for an arbitrary $\phi \in \mathbb{R}^m$, where C_1 is a positive constant in (3.19).

Notice that Assumption [A4]-(i) corresponds to analysis on the transonic flow for the outflow problem of compressible viscous gases. Assumption [A4]-(ii) corresponds to the condition that the Reynolds number is sufficiently small for the model system of compressible viscous gases. The a priori estimate for φ is summarized in the following theorem.

Theorem 3.1. *Assume that Assumptions* [A1], [A2], [A3] *and* [A4] *hold. Let $\tilde{u}$ be a degenerate boundary layer solution obtained in Theorem* 2.2 *and let $\varphi \in C^0([0,T]; L^2)$ be a solution to* (3.1)–(3.3) *for a certain $T > 0$. Then there exists a positive constant ε_0 such that if $\|\varphi_0\|_{L^2} + \delta \le \varepsilon_0$, then φ satisfies the following uniform estimate*

$$\|\varphi(t)\|_{L^2}^2 + \int_0^t \|\varphi_x(\tau)\|_{L^2}^2\, d\tau \le C\|\varphi_0\|_{L^2}^2 \quad (0 \le t \le T). \tag{3.4}$$

Remark. By combining the existence of the solution locally in time with the a priori estimate (3.4), we can construct a solution $\varphi \in C^0([0,\infty); L^2)$ globally in time. Moreover, if we construct the solution in H^1 framework, we can show the asymptotic stability $\|\varphi(t)\|_{L^\infty} \to 0$ as $t \to \infty$. In this paper, we only give the derivation of the basic estimate (3.4) in L^2 framework.

To obtain the estimate (3.4), it is convenient to define the weighted L^2 norm

$$|\varphi|_\alpha := \Big(\int_{\mathbb{R}_+} z^{-\alpha}|\varphi|^2\, dx\Big)^{1/2},$$

where $z(x) > 0$ is a solution to (2.7) satisfying (2.9).

Proof of Theorem 3.1. We employ the energy form $\mathscr{E}$ and the energy flux $\mathscr{F}$ defined by

$$\mathscr{E} = \frac{1}{2}|\varphi|^2, \quad \mathscr{F} = \frac{1}{2}\mathbf{A}[\varphi,\varphi] + \frac{1}{2}\mathbf{F}[\tilde{u},\varphi,\varphi] + \frac{1}{3}\mathbf{F}[\varphi,\varphi,\varphi],$$

$$\mathbf{A}[u,v] := \langle Au, v\rangle = \sum_{i,j} a_{ij}u_i v_j, \quad \mathbf{F}[u,v,w] := \langle u, F(v,w)\rangle = \sum_{i,j,k} f_{ij}^k u_i v_j w_k,$$

where $u = {}^t(u_1, \ldots, u_m)$, $v = {}^t(v_1, \ldots, v_m)$ and $w = {}^t(w_1, \ldots, w_m)$. Note that $\mathbf{A}[u, v]$ and $\mathbf{F}[u, v, w]$ are multi-linear forms. Then we see that $\mathscr{E}$ and $\mathscr{F}$ satisfy

$$\mathscr{E}_t + \mathscr{F}_x + \mathscr{G} + \langle B\varphi_x, \varphi_x \rangle = (\langle B\varphi_x, \varphi \rangle)_x, \tag{3.5}$$
$$\mathscr{G} := \langle \tilde{u}_x, f(u) - f(\tilde{u}) - D_u f(\tilde{u})\varphi \rangle = \frac{1}{2}\mathbf{F}[\tilde{u}_x, \varphi, \varphi].$$

We firstly show the proof of (3.4) under the assumption [A4]-(i) by following the idea in [7]. We change the variable φ to ψ defined by $\psi(t, x) := P^{-1}\varphi(t, x)$ where P is a diagonalization matrix of $B^{-1}A$ satisfying (2.5). Then we see

$$\varphi = P\psi = r\psi_1 + P_*\psi_*, \tag{3.6}$$
$$\psi = \begin{pmatrix} \psi_1 \\ \psi_* \end{pmatrix}, \quad \psi_1(t, x) \in \mathbb{R}, \quad \psi_*(t, x) \in \mathbb{R}^{m-1}.$$

By using ${}^tP = P^{-1}B^{-1}$, we see

$$\mathbf{A}[\varphi, \varphi] = \langle AP\psi, P\psi \rangle = \langle {}^tPAP\psi, \psi \rangle = \langle P^{-1}B^{-1}AP\psi, \psi \rangle = \langle \Lambda\psi_*, \psi_* \rangle.$$

Also, (2.10), (2.11), (3.6) and multi-linearity of $\mathbf{F}$ yield

$$\mathscr{F} = \frac{1}{2}\langle \Lambda\psi_*, \psi_* \rangle + O(|\varphi|^3 + \delta e^{-cx}|\varphi|^2) \le -c_1|\psi_*|^2 + O(|\varphi|^3 + \delta e^{-cx}|\varphi|^2), \tag{3.7}$$
$$\mathscr{G} = \kappa^2 z^2 \psi_1^2 + z^2\mathscr{G}' + O(z^3|\varphi|^2 + \delta e^{-cx}|\varphi|^2), \tag{3.8}$$

where c_1 is a positive constant. Here $\mathscr{G}'$ is a quadratic form consisting of ψ_1 and ψ_* satisfying

$$|\mathscr{G}'| \le C_1(|\psi_1||\psi_*| + |\psi_*|^2),$$

where C_1 is a positive constant. To obtain the estimate (3.4), we employ the weighted energy method with using a weight function

$$W(x) = \frac{\omega}{\omega - \kappa z(x)}, \quad \omega := \frac{2c_1\kappa^4}{9(3C_1^2 + 2C_1\kappa^2)}. \tag{3.9}$$

If δ is small enough to satisfy $|z(x)| \le \omega/2$, we see that W satisfies

$$\frac{2}{3} \le W(x) \le 2, \quad W_x(x) = \frac{1}{\omega}\kappa^2 W^2 z^2 + O(z^3) > 0. \tag{3.10}$$

Multiplying (3.5) by W, we get

$$(W\mathscr{E})_t + (W\mathscr{F})_x - W_x\mathscr{F} + W\mathscr{G} + W\langle B\varphi_x, \varphi_x \rangle = (W\langle B\varphi_x, \varphi \rangle)_x - W_x\langle B\varphi_x, \varphi \rangle. \tag{3.11}$$

We estimate $-W_x\mathscr{F} + W\mathscr{G}$ in (3.11). From (3.7) and (3.10), we have

$$-W_x\mathscr{F} \ge \frac{4\kappa^2 c_1}{9\omega} z^2|\psi_*|^2 + O(z^2|\varphi|^3 + \delta e^{-cx}|\varphi|^2). \tag{3.12}$$

Also, (3.8) and (3.10) give

$$W\mathscr{G} \ge \frac{2\kappa^2}{3} z^2\psi_1^2 - 2C_1 z^2(|\psi_1||\psi_*| + |\psi_*|^2) + O(z^3|\psi|^2 + \delta e^{-cx}|\varphi|^2)$$
$$\ge \frac{\kappa^2}{3} z^2\psi_1^2 - \frac{3C_1^2 + 2C_1\kappa^2}{\kappa^2} z^2|\psi_*|^2 + O(z^3|\psi|^2 + \delta e^{-cx}|\varphi|^2), \tag{3.13}$$

where we have used $2C_1|\psi_1||\psi_*| \leq \frac{\kappa^2}{3}\psi_1^2 + \frac{3C_1^2}{\kappa^2}|\psi_*|^2$. Therefore, (3.12) and (3.13) yield

$$-W_x\mathscr{F} + W\mathscr{G} \geq cz^2|\varphi|^2 + O(z^3|\psi|^2 + z^2|\psi|^3 + \delta e^{-cx}|\varphi|^2). \tag{3.14}$$

The last term in the right-hand side of (3.11) is estimated as

$$|W_x\langle B\varphi_x, \varphi\rangle| \leq C\delta(z^2|\varphi|^2 + |\varphi_x|^2). \tag{3.15}$$

Integrating (3.11) over $[0, T] \times \mathbb{R}_+$, substituting (3.14) and (3.15) in the resultant equality and letting δ suitable small, we have

$$\|\varphi\|_{L^2}^2 + \int_0^t (|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2)\, d\tau \leq C\|\varphi_0\|_{L^2}^2 + C\int_0^t \int_{\mathbb{R}_+} (z^2|\varphi|^3 + \delta e^{-cx}|\varphi|^2)\, dx\, d\tau. \tag{3.16}$$

To estimate the remainder terms in the right-hand side, we compute

$$\int_{\mathbb{R}_+} z^2|\varphi|^3\, dx \leq \|\varphi\|_{L^\infty}|\varphi|_{-2}^2 \leq C\|\varphi\|_{L^2}(|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2), \tag{3.17}$$

where we have used the Sobolev inequality $\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{L^2}^{1/2}\|\varphi_x\|_{L^2}^{1/2}$ and $|\varphi|_{-2} \leq C\|\varphi\|_{L^2}$. Also, due to the Poincaré type inequality, we see

$$\int_{\mathbb{R}_+} e^{-cx}|\varphi|^2\, dx \leq C\|\varphi_x\|_{L^2}^2. \tag{3.18}$$

Substituting (3.17) and (3.18) in (3.16), we get

$$\|\varphi\|_{L^2}^2 + \int_0^t (|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2)\, d\tau \leq C\|\varphi_0\|_{L^2}^2 + C\sup_{0\leq t\leq T}\|\varphi\|_{L^2}\int_0^t (|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2)\, d\tau$$

which yields the desired estimate (3.4) provided that $\|\varphi_0\|_{L^2}$ is sufficiently small.

Next, we show the proof of (3.4) under Assumption [A4]-(ii). By using

$$|\mathscr{G}| \leq \frac{1}{2}|\tilde{u}_x||F(\varphi, \varphi)| \leq C\tilde{F}z^2|\varphi|^2 \leq C\tilde{F}\frac{1}{x^2}|\varphi|^2$$

and the Hardy inequality, we see

$$\int_{\mathbb{R}_+} |\mathscr{G}|\, dx \leq C\tilde{F}\int_{\mathbb{R}_+} \frac{1}{x^2}|\varphi|^2\, dx \leq C_1\tilde{F}\|\varphi_x\|_{L^2}^2, \tag{3.19}$$

where C_1 is a positive constant. Thus, integrating (3.5) and substituting the above inequality with using Assumption [A4]-(ii), we obtain the desired estimate (3.4). Consequently, we complete the proof. □

Notice that the computation in the present paper is also applicable to the model system of compressible and viscous gas which is given by a hyperbolic-parabolic system. We also note that the condition [A4] is assumed because of the technical reason. It is open problem that we can remove this conditionor not.

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Conflict of Interest

The author declare no conflicts of interest in this paper.

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A survey of critical structures in competitive games

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A B S T R A C T

One of the biggest problems of human society is facing crises. Origins of many crises go back to strategy selection in the relations between human beings. The international community is facedwith many crises, such as poverty and lack of development of a large section of human society, globalwarming, economic crises, the incidence of infectious diseases, the accumulation of weapons of massdestruction, wars, migration, lack of food and clean drinking water are among the crises that threateninternational community. Each of these challenges alone would require measures and facilities that inmany cases are beyond the limited resources of the international community. In this article, the criseshave been discussed, whose origin is relations between human beings. By defining critical points in2 × 2 games, we provide a mathematical model to detect this type of crises, and then by defining aunique compromise point, we offer solutions for this type of crisis. Sometimes the compromise pointcorresponds to the Nash equilibrium, and sometimes better than Nash equilibrium. We believe thatwhat is presented in this article can help fill the void. Fixing the vacuum in game theory and optimaluse of compromise and critical points leads to the development of cooperation–cooperation strategy inthe world.

Keywords: critical point; compromise point; Cuban Missile Crisis; cooperation strategy;non-cooperative games

1. Introduction

The international community is faced with many crises, such as poverty and lack of development of a large section of human society, global warming, economic crises, the incidence of infectiousdiseases, the accumulation of weapons of mass destruction, wars, migration, lack of food and cleandrinking water are among the crises that threaten international community. The other considerablecrises that pose new problems for the international community, such as increase in spending on arms rise in refugees to Europe, increased hunger in developing countries and environmental crises.

The international community and non-governmental international institutions active in disarmament and arms control have focused their attention to the crisis of rising costs of weapons of mass destruction because of the destructive power of weapons of mass destruction, especially nuclear weapons. The deteriorating situation in this area represents a major crisis in the international community that we callit crisis of confidence. Crisis of confidence opens the way for irrational processes. The economiccrises have exacerbated this, so that large arm factories, mainly owned by the powerful countries, lookfor an arena to transfer and stockpile weapons widely and publicly. In fact, the issue of the transfer ofconventional weapons is the reason behind some global conflicts and undermines international stabilityand security. This is while there are no mechanisms to control these weapons.

Military spending in 1970 was equivalent to 235 billion dollars and in 1985 reached about 940 billion dollars. The costs in 2002 reached its lowest level, but since 2002, this figure has been risingagain. In 2008, the figure was beyond one trillion, four hundred and sixty-four billion dollars. Thistrend has continued until the arm cost of the first 10 countries in this regard (America, China, Russia,Saudi Arabia, etc.) has reached the figure over a trillion four hundred billion dollars [30, 31, 32, 33].The crisis of immigration to Europe reached its highest in 2015 with an increase in the numberof asylum seekers and economic migrants from regions like the Middle East (Syria Iraq, Palestine(Africa) Eritrea, Mali, Kambiya, Somalia (Balkans), Albania, Kosovo, Monet Montenegro, Bosniaand Herzegovina, Serbia (And South Asia), mostly from Afghanistan, Pakistan and Bangladesh goingto European Union through Southeast Europe and the Mediterranean. According to United NationsHigh Commissioner for Refugees, by the end of August 2015, seventy percent of refugees were fromSyria, Afghanistan and Eritrea. The term refugee crisis became prevalent in April 2015 followingthe sinking of five boats carrying two thousands of refugees to Europe on the Mediterranean Sea andkilling more than 1,200 people [3].

Another crisis in the twenty-first century making the international community suffer is hunger crisis. The main cause of poverty and hunger in the twenty-first century is unfair global economic and politicalsystems. In addition, a minority group usually monopolizes control over resources and earnings powerbased on military, political, and economic issues and lower classes of society get less of them. Wars arean important factor in the spread of hunger and poverty. Climate changes are known as an influentialfactor in the spread of hunger and poverty. Increased droughts, floods, changing weather patterns havenegative effects on agricultural work and lives of people around the world. According to FAO, nowclose to 870 million from 7.1 Billion people of the world, i.e. one eighth of the world's populationsuffers from chronic malnutrition.

Almost all the hungry people live in the developing countries. The number of undernourished people in the Asia-Pacific has reduced to 563 million from 739 million reduced by 30 percent. In LatinAmerica and the Caribbean, 65 million hungry people in 1990–1992 have reduced to 49 Million in2010–2012. However, the number of hungry people in Africa has increased from 175 million to 239 million people i.e. one out of four in Africa are hungry [12].

What we are seeing now, is the results of hundreds of years of unequal development in the rich world that passed the vast majority of other countries in the world. Therefore, the people not includedin this development look for a better life, and this determination has placed disproportionate burden onthe boundaries between the world's rich and poor. Poverty reduction in poor countries will solve theproblem of refugees, but this will not happen quickly. In the short term, the stabilization of unstable political situation in conflict zones would help [4, 5, 6, 11].

When speaking of the crisis, we must define crisis in accordance with the conditions of the peo ple involved with it. A critical moment is the turning point for better or worse life that is a short but meaningful definition. In general, it should be accepted that offering a clear definition of crisis is verydifficult and all definitions are relative. This is because a subject may be a crisis to an individual, orga nization or society, but not for the other. However, the fact that in critical situations something urgentand serious must be done for the condition not to get more critical is acceptable to all communities.Some crises arise suddenly and abruptly and have sudden effects on the internal and external environ ment of the organization. These crises are called sudden crises. On the contrary, there are gradualor density crises that start from a series of critical issues and are strengthened over time, continue toa threshold level, and then arise. From the perspective of Parsons, sudden crisis will be gradual andcontinuous. Sudden crises have no prior warning signs and organizations are not able to investigatethem and plan to do away with them. Crises created gradually and slowly can be stopped or restrictedby organizational measures. Continues crises may last weeks, months or even years. Strategies to dealwith these crises in different situations depend on the time pressure and the extent of control of theseevents. Mitraf uses two spectra for the classification of crises. One spectrum determines the crisesbeing external or internal: whether crises happen within or outside the organization. Other spectrumdetermines crises being technical or social.

The first division of crisis can be individual, group, organizational, and social. Social crises are divided into political, cultural, economic, health, natural or a combination of these crises. Usually it isthought that only social crises should be managed, but the fact is that social crisis must be managed first. Facts and figures such as population growth rate, age composition of the population, unemployment rate, growing curve of industry, growth rate, the percentage of dropout at different levels, the capacity of accepting technical and vocational education, the growth rate of some diseases, the growth rate ofaddiction, suicide rates in the age groups and social status and gender, and many simple statisticalresults on the one hand show a very special circumstance, and the other hand, represent the inevitablenecessity of knowledge management in the public service and management.Causes of the crisis are very different. A psychological variable, a sudden attack, diplomatic ten sions, war, coup, collapse of states, states of turmoil, violent protests, ethnic conflicts, the studentmovement, non-regulatory challenges of political factions, and so on each one can be a severe anddestructive source of crisis.

2. Materials and method

2.1. Game theory

Definition : (Nash equilibrium) The action profile $a*$ in a strategic game with ordinal preferences is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j . Equivalently, for every player i,

$$u_i(a*) \geq u_i(a_i, a^*_{-i}) \text{ for every action } a_i \text{ of player } i$$

Where u_i is a payoff function that represents player i's preferences [26].

Whenever there are several Nash equilibriums in a game and the players have to choose the same strategy, if they are wise, they must find a way to coordinate their beliefs and expectations concerning choice and practice of each other. One of these methods is "focal point". The influential element in convergence of expectations and beliefs depends on culture, rituals, and customs. Thomas Schelling f irst presented the idea in 1969. In his opinion focal point or focal points for each person mean his expectations about others' expectations of his expectations. In this article, we will take a new approach in GT to solve many crises in modern societies: crises that GT has not provided any solutions. For example, in chicken game, GT finds a strategy that leads to Nash equilibrium and the most important goal of approaches Nash equilibrium is when one party gives up the other continues. We shall show that in chicken game, there is a better Nash equilibrium point that it is withdrawing from the game by both players, which is called compromise in this article. To illustrate the importance of this new definition, we refer to the famous of prisoner's dilemma game of two prisoners [23, 25, 27, 34] that show Nash equilibrium is not necessarily the best choice and this is when both prisoners choose to "compromise" and get the best out of this collaboration.

2.1.1. Crisis in game theory

Crisis in non-cooperative games: the process or change that disrupts the balance (balances) of the target population gets the community outside of normal state, and takes it towards border of cooperation–non-cooperation (non-cooperation–non-cooperation) is called crisis. Depending on the definition of crisis and crisis community, crisis can be classified in di erent ways. Here, we consider the new categorizing of crises as a whole:

(1) Man-made crises- natural disasters
(2) Predictable crises- unpredictable crises
(3) Controllable crisis- uncontrollable crises
(4) Immediate crises- crises over time

Our research is on man-made, predictable, controllable uncontrollable, and over time crises. Our goal is to identify critical points by modeling the structure of the crisis in the community in GT. In the future, with the help of time series and random process, we will have o ered statistical models that using the roots of the crisis will have the power of forecasting crisis over time and then we can obtain the target population crisis. If the crisis is controllable, in the stage before crisis, we control it, and if it is uncontrollable, with the predictions and with the help of crisis management, we provide the ground to minimize the e ects of crisis while happening and the consequences after it.

Crisis point: in the game G, if N is the number of players and S_i is set of strategies for player i. The payo of the show player i with u_i

$$u_i : S \rightarrow R \qquad \forall i \in N.$$

$\mathcal{S}$ Cartesian product strategy players: $\mathcal{S} = S_1 \times S_2 \times S_3 \times \times S_n$
For example, the payoff of the players the strategy $(s_1, s_1, , s_1)$ is defined as follows:

$$u_1(a^1) = u_1(s_1, s_1, , s_1) \in R$$

$$u_2(a^1) = u_2(s_1, s_1, , s_1) \in R$$

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$$u_n(a^1) = u_1(s_1, s_1, , s_1) \in R$$

According to the assumptions of the point (points) crisis are defined as follows:

$$C_i(a^k) = (\ u_i(a^k)\ ,\ u_{-i}(a^k)\ ) = \exists i \in \mathbb{N}, \forall k \begin{cases} u_i(a^*) = Max\left\{u_i(a^k)\ :\ a^k \in A_i\right\} \\ and \\ \forall j, j \neq i\ \ u_j(a^*) = Min\left\{u_j(a^k)\ :\ a^k \in A_i\right\} \end{cases}$$

In this case, K is the player crisis making, and $N - K$ is the player crisis-stricken. This kind of crisis is called the first type of crisis. If

$$C_i(a^k) = (\ u_i(a^k)\ ,\ u_{-i}(a^k)\ ) = \exists i \in \mathbb{N}, \forall k \begin{cases} u_i(a^*) = Min\left\{u_i(a^k)\ :\ a^k \in A_i\right\} \\ and \\ \forall j,\ j \neq i\ \ u_j(a^*) = Min\left\{u_j(a^k)\ :\ a^k \in A_i\right\} \end{cases}$$

Here are all the players are crisis-stricken. This kind of crisis is called the second type of crisis.

In the game G, if $N = \{1, 2\}$ number of players, and $S_1 = \{s_1, s_2\}$, $S_2 = \{s_1, s_2\}$ strategy players. u_i payoff of player i is

$$u_i : S \rightarrow R \qquad \forall i \in N.$$

$\mathcal{S}$ Cartesian product strategy players:

$$\mathcal{S} = S_1 \times S_2 = \{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}$$

Payoff of players for each combination of strategies are defined as follows:

$$u_1(a^1) = u_1(s_1, s_1) \in R \quad , \quad u_2(a^1) = u_2(s_1, s_1) \in R$$

$$u_1(a^2) = u_1(s_1, s_2) \in R \quad , \quad u_2(a^2) = u_2(s_1, s_2) \in R$$

$$u_1(a^3) = u_1(s_2, s_1) \in R \quad , \quad u_2(a^3) = u_2(s_2, s_1) \in R$$

$$u_1(a^4) = u_1(s_2, s_2) \in R \quad , \quad u_2(a^4) = u_2(s_2, s_2) \in R$$

According to the assumptions of the point (points) crisis are defined as follows:

$$C_i(a^j) = (\ u_i(a^j)\ ,\ u_{-i}(a^j)\ ) = \exists i \in \mathbb{N}, \forall k \begin{cases} u_i(a^*) = Max\left\{u_i(a^k)\ :\ a^k \in A_i\right\} \\ and \\ \forall j,\ j \neq i\ \ u_j(a^*) = Min\left\{u_j(a^k)\ :\ a^k \in A_i\right\} \end{cases}$$

Or

$$C_i(a^j) = (\ u_i(a^j)\ ,\ u_{-i}(a^j)\ ) = \exists i \in \mathbb{N}, \forall k \begin{cases} u_i(a^*) = Min\left\{u_i(a^k)\ :\ a^k \in A_i\right\} \\ and \\ \forall j,\ j \neq i\ \ u_j(a^*) = Min\left\{u_j(a^k)\ :\ a^k \in A_i\right\} \end{cases}$$

In each game, according to the preferences of the players, point of cooperation–cooperation is the compromise point of the game. Compromise point is used to resolve the crisis in the game. Compromise point is a unique in the 2 2 games. Because in the 2 2 games, there is just a house of matrix games that both players choose to cooperate. In some of these games, this point overlaps with Nash equilibrium, and in some games, there is a better choice for players than Nash equilibrium. By studying the structure of crises, both natural and unnatural, we concluded that crisis and crisis making are in context of games. In other words, we can show with what strategies players create crisis and what the best way to deal with it is what strategy. According to the terms of the game and the preferences of the players, we can define a crisis point in the 2 2 games. Interestingly Stag Hunt game does not have a point of crisis. Stage hunt game is based on cooperation, bilateral trust and patience is built and players who choose to play Stage hunt are aimed at cooperation–cooperation. Choosing strategy in Stage hunt at first glance is very simple. The result of cooperation is more fruitful than fraud (in the language of game theory, betrayal), so we should always consider cooperation and get better results. This is opposite the prisoner's dilemma. This dilemma stems from the fact that regardless of the actions of the other side of the game, the result is always to the benefit of dishonest person. However, what is problematic in stage hunt game is the element of risk. Accordingly, it is clear that in this game there is no crisis.

So far, in GT, Nash equilibrium has represented an unchangeable point for the players, in which collective profit has had priority over individual profit, and at the mentioned point, none of the players want to change their strategy. In this article, we show that there is sometimes a better choice for players than Nash equilibrium. The critical point in each game represents the worst and the most selfish choice for a player and shows that if players choose this point, sometimes they themselves, and sometimes other players incur the lowest possible impact on the game. To compensate for this, the best strategy for players, according to their strategy preferences, is to trust each other and cooperationcooperation. By recognizing the critical point and the point of compromise in game, one can move players in the direction that they adopt strategy of cooperation–cooperation and trusting each other in the first iteration of the game [21, 22, 24, 26, 34].

2.1.2. Prisoner's dilemma games

The prisoner's dilemma game [26, 34] is based on a lack of trust in the opponent and shows the state where without trusting the opponent players cannot gain more, and in the best state gain Nash equilibrium in GT. While in this game, there is a better option to choose. Recognizing the crisis points of the game and then identifying points of compromise of the game make players achieve cooperationcooperation with one iteration of strategy.

	C	D
C	R, R	S, T
D	T, S	P, P

$$T > R > P > S$$

Players payoff will be as follows:

$$u_1(a^1) = u_1(C,C) = R \in R \quad , \quad u_2(a^1) = u_2(C,C) = R \in R$$

$$u_1(a^2) = u_1(C,D) = S \in R \quad , \quad u_2(a^2) = u_2(C,D) = T \in R$$

$$u_1(a^3) = u_1(D,C) = T \in R \quad , \quad u_2(a^3) = u_2(D,C) = S \in R$$

$$u_1(a^4) = u_1(D,D) = P \in R \quad , \quad u_2(a^4) = u_2(D,D) = P \in R$$

In the prisoner's dilemma game when crisis occurs when one of the players pursue cooperation, and another defect. The points (CD) and (DC), are critical points.

$$C_1(a^2) = (\ u_1(a^2)\ ,\ u_2(a^2)\ )\ =\ (C,D) = \left\{\forall l = 1,3,4\ \ u_1(a^2)\ \leq\ u_1(a^l)\ \ \&\ \ u_2(a^2)\ \geq\ u_2(a^l)\right\}$$

$$C_2(a^3) = (\ u_1(a^3)\ ,\ u_2(a^3)\ )\ =\ (D,C) = \left\{\forall l = 1,2,4\ \ u_1(a^3)\ \geq\ u_1(a^l)\ \ \&\ \ u_2(a^3)\ \leq\ u_2(a^l)\right\}$$

Critical points of the game, the crisis of the first kind. In other words, the min and max payo for the players. In this game the best choice against the crisis, choose a point of compromise. This is a strategy of cooperation–cooperation (CC). It should be noted that in the prisoner's dilemma game, Nash equilibrium can also help to resolve the crisis in the long time. But the compromise, better and more appropriate way. Indeed, if we use the Nash equilibrium to solve the crisis, there is the possibility that players will move towards the crisis point. In this case, the game has to be repeated several times so players go to the compromise point and the crisis will be resolved. As a result, in the prisoner's dilemma game, choosing a compromise point is better than Nash equilibrium. In the future, with a focus on a compromise point, perhaps a good solution could be found to counter the ZD strategy.

2.1.3. Chicken game

Returning to one-on-one situations, we come to the dangerous game of Chicken. Here it is not as much a matter of assigning specific numerical values to rewards (which can be di cult in many cases) as of looking at how well you might do out of a situation in the order: good, neutral, bad, worst [33]. The structure is designed to start the chicken game in such a crisis. In this game Hawk and Dove to take a prey to compete. Each strategy ahead of them.

	Hawk	Dove
Hawk	X, X	W, L
Dove	L, W	T, T

$$W > T > L > X$$

$$u_1(a^1) = u_1(Hawk, Hawk) = X \in R \quad , \quad u_2(a^1) = u_2(Hawk, Hawk) = X \in R$$

$$u_1(a^2) = u_1(Hawk, Dove) = W \in R \quad , \quad u_2(a^2) = u_2(Hawk, Dove) = L \in R$$

$$u_1(a^3) = u_1(Dove, Hawk) = L \in R \quad , \quad u_2(a^3) = u_2(Dove, Hawk) = W \in R$$

$$u_1(a^4) = u_1(Dove, Dove) = T \in R \quad , \quad u_2(a^4) = u_2(Dove, Dove) = T \in R$$

Nashequilibria are (Hawk, Dove) (Dove, Hawk). Thecritical point is Game(Hawk, Hawk)because players with a choice of strategy of non-cooperation–non-cooperation to achieve the worst possible outcome. The crisis of the second type and the two rivals are crisis-stricken. In fact, min and min consequences of two players. In interpreting this game if two hawk to seize prey heavily collided with each other to create a crisis where they may both be killed.

$$C_1(a^1) = (\, u_1(a^1), u_2(a^1)\,) \;=\; (X, X) \;= \left\{\forall l = 2,3,4 \;\; u_1(a^1) \le u_1(a^l) \;\;\&\;\; u_2(a^1) \le u_2(a^l)\right\}$$

Or equivalent

$$C_2(a^1) = (\, u_1(a^1), u_2(a^1)\,) \;=\; (X, X) = \left\{\forall l = 2,3,4 \;\; u_1(a^1) \le u_1(a^l) \;\;\&\;\; u_2(a^1) \le u_2(a^l)\right\}$$

In this game, Nash equilibrium cannot be one way to resolve the crisis because the lack of cooper ation by one of the players may increase the severity of the crisis. The only and best solution in this game is a point of compromise that (Dove, Dove). It was used to solve the Cuban missile crisis from a compromise point. If they were using Nash equilibrium, disaster would occur. In this game, the compromise point is absolutely superior to Nash equilibrium.

3. Conclusions and discussion

Crisis of irrigation systems: in the article evolution of game theory application in irrigation sys tems, conflicts and crises emerged over the use of water and irrigation, and put them analyzed by game theory. The first recorded dispute in antiquity took place between the cities of Umma and Lagash in the Middle East over irrigation systems and diversion of water from Tigris and Emphratis rivers. That dispute had lasted for 100 years from 2500 to 2400 B.C. Continuing conflicts over Mesopotamia through passing of years led Hammurabi the king of ancient Babylon in 1790 B.C. to enforce laws prohibiting water theft in irrigation systems, in his famous "Hammurabi's Code" [25]. Crisis and conflicts for water between cities in the value of farmland in eastern California in the nineteenth century, successive conflicts over water rights between India and Pakistan to the brink of war went ahead, the fight over the Jordan River Jordan, Russia and Israel in the 1950s and 1960s, and... All these are examples of crisis and conflicts over water and irrigation in the world that the structure of game theory, the prisoner's dilemma game has been analyzed.

One of the games mentioned in this article groundwater pumping game that was introduced by Madden(2010). Inthis game, players are going to use the rationality game, to perform non-cooperation with each other. The structure of the game, when the crisis will occur when a farmer PRL strategy and other strategies to adopt PRH and a conflict arises between farmers.

$$C_i(a_{-i}) = (\,(PRL, PRH)\,,\ (PRH, PRL)\,)$$

Farmer who chooses PRL strategy is the crisis and by Strategy PRH the farm is crisis-making. There is a better way for the farmer to solve this conflict Nash equilibrium can use it and it is this strategy (cooperation–cooperation) or the same (PRLPRL) [25].

Cuban missile crisis: another application of the critical point, which implies the importance and strength of this point, see the article "A Game Theoretic History of the Cuban Missile Crisis".

This study surveys and evaluates previous attempts to use game theory to explain the strategic dy namic of the Cuban missile crisis, including, but not limited to, explanations developed in the style of Thomas Schelling, Nigel Howard and Steven Brams [2]. And shows the existing vacuum is triggered, the Cuban missile crisis by game theory is not well analyzed. A crisis that has been characterized, without exaggeration, as the "defining event of the nuclear age". All of the explanations were judged to be either incomplete or deficient in some way. Schelling's explanation is both empirically and the oretically inconsistent with the consensus interpretation of the crisis; Howard's with the contemporary understanding of rational strategic behavior; and Brams' with the full sweep of the events that define the crisis. Equally troubling is the scant empirical evidence that the Kennedy administration either manipulated the risk of war during the crisis with "mathematical precision", as Schlesinger and some other insider accounts have claimed, or successfully made use of any related brinkmanship tactics that resulted in a clear US victory. The crisis ended only when both sides "blinked". Nigel Howard's meta-game analysis of the missile crisis also fails to provide a compelling explanation. Similarly, the improved meta-game technique of Fraser and Hipel falls short of the explanatory mark. Like Howard [14], Fraser and Hipel find that the compromise outcome is an equilibrium in their dynamic model, but are unable to explain, at least game-theoretically, why it, and not another co-existing equilibrium, ended the crisis [2, 13, 15].

Problem in their conclusions of their analysis on game theory, but there is a major vacuum in the game theory. The vacuum in the Cuban missile crisis as a critical point of chicken game, if occurred, would start a nuclear war in the world and both games were in crisis-stricken. The research was carried out by game theory, the two countries to resolve the crisis were to the strategy of cooperationcooperation as the only way to solve the critical point in game theory is the compromise point. In order to solve the Cuban missile crisis selected the cooperation–cooperation as the best option. This is a more appropriate choice of Nash equilibrium in the chicken game.

Zero-determinant strategies, extortion: Recently, Press and Dyson have proposed a new class of probabilistic and conditional strategies for the two-player iterated prisoner's dilemma, so-called zerodeterminant strategies. A player adopting zero-determinant strategies is able to pin the expected payo of the opponents or to enforce a linear relationship between his own payo and the opponents' payo , in a unilateral way. This paper considers zero-determinant strategies in the iterated public goods game, a representative multiplayer game where in each round each player will choose whether or not to put his tokens into a public pot, and the tokens in this pot are multiplied by a factor larger than one and then evenly divided among all players. The analytical and numerical results exhibit a similar yet di erent scenario to the case of two-player games: (i) with small number of players or a small multiplication factor, a player is able to unilaterally pin the expected total payo of all other players; (ii) a player is able to set the ratio between his payo and the total payo of all other players, but this ratio is limited by an upper bound if the multiplication factor exceeds a threshold that depends on the number of players [28].

$$\tilde{p} \equiv (-1 + p_1, -1 + p_2, p_3, p_4)$$

Is solely under the control of X; whose third column,

$$\tilde{q} \equiv (-1 + q_1, q_3, -1 + q_2, q_4)$$

Is solely under the control of *Y*; and whose fourth column is simply *f*. *X*'s payo matrix is $S_X = (R,S,T,P)$whereas *Y*'s is $S_Y = (R,T,S,P)$[29].

According to the preferences of the players see the player *X*, the consequences of player *Y* holds and adjusts the moving with preferences prisoner's dilemma game. But the player *Y* with changes in preferences so that he knows the plays during the game, the best strategy is cooperation and will receive the greatest consequence of (*T*) may be in play. However, in contrast to cooperate with defect in prisoner's dilemma games cooperation strategy will get the lowest payo (S). In fact, this is critical point in game and player *X* takes the control of the game with using the critical points in the prisoner's dilemma game and strategy pin [1, 16, 17, 18, 19, 20, 28, 29]. In this case, player *X* is making crisis and player *Y* is crisis-stricken. Player *X* due to the crisis that led to his opponent could extortion him. What is surprising is not that *Y* can, with *X*'s connivance, achieve scores in this range, but that *X* can force any particular score by a fixed strategy *p*, independent of *Y*'s strategy q. In other words, there is no need for *X* to react to *Y*, except on a timescale of her own choosing. *A* consequence is that *X* can simulate or "spoof" any desired fitness landscape for *Y* that she wants, thereby guiding his evolutionary path [29].

The question is whether the crisis point we can say there is a ZD strategy for all 2 2 games? If established, would follow this structure?

Given the widerange ofgametheoryinvarious fields of political, economic, social and international relations, the question raised here is whether game theory is its ability to be as dynamic systems in medical sciences, particularly in the field of used to predict disease?

Can the evolutionary stable strategy (ESS) used in order to prevent the spread of communicable diseases such as Ebola, Zika and types of flu?

Conflict of interest

The authors declare no conflicts of interest in this paper.

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2 Conference papers may only be submitted if the paper has been completely re-written (taken to mean more than 50%) and the author has cleared any necessary permission with the copyright owner if it has been previously copyrighted.

3 All our articles are refereed through a double-blind process.

4 All authors must declare they have read and agreed to the content of the submitted article and must sign a declaration correspond to the originality of the article.

Submission Process

All articles for this journal must be submitted using our online submissions system. http://enrichedpub.com/ . Please use the Submit Your Article link in the Author Service area.

Manuscript Guidelines

The instructions to authors about the article preparation for publication in the Manuscripts are submitted online, through the e-Ur (Electronic editing) system, developed by **Enriched Publications Pvt. Ltd**. The article should contain the abstract with keywords, introduction, body, conclusion, references and the summary in English language (without heading and subheading enumeration). The article length should not exceed 16 pages of A4 paper format.

Title

The title should be informative. It is in both Journal's and author's best interest to use terms suitable. For indexing and word search. If there are no such terms in the title, the author is strongly advised to add a subtitle. The title should be given in English as well. The titles precede the abstract and the summary in an appropriate language.

Letterhead Title

The letterhead title is given at a top of each page for easier identification of article copies in an Electronic form in particular. It contains the author's surname and first name initial .article title, journal title and collation (year, volume, and issue, first and last page). The journal and article titles can be given in a shortened form.

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1. Original scientific paper (giving the previously unpublished results of the author's own research based on management methods).
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The article should be in English. The grammar and style of the article should be of good quality. The systematized text should be without abbreviations (except standard ones). All measurements must be in SI units. The sequence of formulae is denoted in Arabic numerals in parentheses on the right-hand side.

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An abstract is a concise informative presentation of the article content for fast and accurate Evaluation of its relevance. It is both in the Editorial Office's and the author's best interest for an abstract to contain terms often used for indexing and article search. The abstract describes the purpose of the study and the methods, outlines the findings and state the conclusions. A 100- to 250-Word abstract should be placed between the title and the keywords with the body text to follow. Besides an abstract are advised to have a summary in English, at the end of the article, after the Reference list. The summary should be structured and long up to 1/10 of the article length (it is more extensive than the abstract).

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Acknowledgements

The name and the number of the project or programmed within which the article was realized is given in a separate note at the bottom of the first page together with the name of the institution which financially supported the project or programmed.

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All the captions should be in the original language as well as in English, together with the texts in illustrations if possible. Tables are typed in the same style as the text and are denoted by numerals at the top. Photographs and drawings, placed appropriately in the text, should be clear, precise and suitable for reproduction. Drawings should be created in Word or Corel.

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Note