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Aim & Scope

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GEODETIC ECCENTRIC DOMINATION IN GRAPHS

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ABSTRACT

A subset D of the vertex set G (V) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every ,v ∈ V-D there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the eccentric domination number and is denoted by Yed (G). A set S of vertices in a graph G is a geodetic dominating set if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set is the geodetic domination number of G and is denoted by Yg(G). A set S of vertices in a graph G is a geodetic eccentric dominating set if S is both a geodetic set and an eccentric dominating set. The minimum cardinality of a geodetic eccentric dominating set is the geodetic eccentric domination number of G and is denoted by Yged (G). In this paper, we obtain some bounds for Y ged(G). Exact values of Y ged(G) for some particular classes of graphs are obtained. Also, we characterize graphs for which Y ged(G) = 2 , p-1 and p.

Keywords: Domination, Eccentric domination, Geodetic Domination, Geodetic Eccentric Domination.

Introduction

Let G be a finite, simple, connected and undirected (p,q) graph with vertex set *V(G)* and edge set *E(G)* For graph theoretic terminology refer to Harary [17], Buckley and Harary [11]. The concept of domination in graphs is originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. For details on domination theory, refer to Haynes, Hedetniemi, and Slater [18].

Definition 1.1. The distance *d (u, v)* between two vertices u and v in G is the minimum length of a u−v path.

Definition 1.2. Let G be a connected graph and v be a vertex of G. The *eccentricity* $e(v)$ of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $diam(G) = d(G)$ is the maximum eccentricity. For any connected graph G, $r(G) \le diam(G) \le 2r(G)$. The vertex v is a central vertex if $e(v) = r(G)$. The center $C(G)$ is the set of all central vertices. For a vertex v, each vertex at a distance $e(v)$ from v is an eccentric vertex of v. Eccentric vertex set of a vertex v is defined as $E(v) = \{u \in V(G)/d(u, v) = e(v)\}$. The set E_k denotes the set of vertices of G with eccentricity k.

The concept of domination in graphs was introduced by Ore in [21]. In 1977, Cockayne and Hedetniemi explained importance and properties of domination in [15].

Definition 1.3 [15, 18]**.** A set $D \subseteq V$ is said to be a *dominating set* in G if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by $\gamma(G)$.

Chartrand et al. studied the concept of geodetic sets in graphs and on the geodetic number of a graph [12, 13, 14]. They also studied the concept of geodomination in graphs. Escuadro et al. [16] studied the concept of geodetic domination in graphs.

Definition 1.4 [13]. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. The closed interval $I[x, y]$ consists of x, y and all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V(G)$, $I[S] = \bigcup_{x, y \in S} I[x, y]$.

Definition 1.5 [13]**.** A set S of vertices in a graph G is a *geodetic set* if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the geodetic number of G and is denoted by $g(G)$.

Definition 1.6 [16]**.** A set S of vertices in a graph G is a *geodetic dominating set* if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set is the geodetic domination number of G and is denoted by $\gamma_g(G)$.

Janakiraman, Bhanumathi and Muthammai [19] introduced Eccentric domination in Graphs. Bhanumathi and Muthammai studied some bounds for $\gamma_{ed}(G)$ and $\gamma_{ed}(T)$ in [1, 2, 3]. Bhanumathi, John Flavia and Kavitha [4] studied the concept of Restrained Eccentric domination in graphs. Bhanumathi and John Flavia studied the concept of Total Eccentric domination in graphs and some more bounds for $\gamma_{ed}(G)$ in [5, 7]. Bhanumathi and Sudhasenthil [6, 8, 9] studied the concept of the split and Nonsplit Eccentric domination, Distance closed eccentric domination, Eccentric domination and chromatic number in graphs. Bhanumathi and Meenal Abirami [10] studied the concept of Upper Eccentric Domination in Graphs. Geodetic eccentric dominating set was defined by Nishanthi in [20].

Definition 1.7 [19]. A set $D \subseteq V(G)$ is an *eccentric dominating set* if D is a *dominating set* of G and for every $v \in V - D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the *eccentric domination number* and is denoted by $\gamma_{ed}(G)$.

Let S be a vertex set of G. Then S is known as an eccentric vertex set of G if for every v is belongs to $V - S$, S has at least one vertex u such that vertex u belongs to eccentric vertex set $E(v)$. An eccentric vertex set S of G is a minimal eccentric vertex set if no proper subset S' of S is an eccentric vertex set of G. S is known as a minimum eccentric vertex set if S is an eccentric vertex set with minimum cardinality. Let $e(G)$ be the cardinality of a minimum eccentric vertex set of G, $e(G)$ is known as eccentric number of G.

Theorem 1.1 [15]. *For any graph* G, $\lceil p/(1 + \Delta(G)) \rceil \leq \gamma(G) \leq p - \Delta(G)$.

Theorem 1.2 [16]. (i) $\gamma_g(K_{1,n}) = n$.

(ii) $\gamma_g(K_{m,n}) = \min\{m, n, 4\}$ *for* $m, n \geq 2$.

Theorem 1.3 [16]. *If* G *is a connected graph of order* $p \geq 2$, *then* $2 \leq \max\{g(G), r(G)\} \leq \gamma_g(G) \leq p$.

Theorem 1.4 [16]. *Let* G *be a connected graph of order* $p \geq 2$. *Then.*

(a) $\gamma_g(G) = 2$ *if and only if there exists a geodetic set* $S = \{u, v\}$ *of* G *such that* $d(u, v) \leq 3$.

(b) $\gamma_g(G) = p$ *if and only if* G *is the complete graph on* p *vertices.*

(c) $\gamma_g(G) = p - 1$ *if and only if there is a vertex* v *in* G *such that* v *is adjacent to every other vertex of* G *and* $G - v$ *is the union of at least two complete graphs.*

Theorem 1.5 [16]. *If* G *is a connected graph with* $\gamma_g(G) = 1$, *then* $\gamma_g(G) = g(G)$.

Theorem 1.6 [19]. (i) $\gamma_{ed}(G) = 1$ *if and only if* $G = K_p$.

(ii) $\gamma_{ed}(K_{1,n}) = 2$, $n \geq 2$.

(iii) $\gamma_{ed}(K_{m,n}) = 2$.

(iv) $\gamma_{ed}(W_3) = 1$, $\gamma_{ed}(W_4) = 2$, $\gamma_{ed}(W_n) = 3$ *for* $n = 5$, $\gamma_{ed}(W_6) = 2$, $\gamma_{ed}(W_n) = 3$ *for* $n \geq 7$.

(v) $\gamma_{ed}(P_p) = \lceil p/3 \rceil$ *if* $p = 3k + 1$

$\gamma_{ed}(P_p) = \lceil p/3 \rceil + 1$ *if* $p = 3k$ *or* $3k + 2$

(vi) $\gamma_{ed}(C_p) = p/2$ *if p is even.*

$\gamma_{ed}(C_p) = \lceil p/3 \rceil$ or $\lceil p/3 \rceil + 1$ *if p is odd.*

In [22] Sudhasenthil proved the following theorem.

Theorem 1.7 [22]. *Let G be a connected graph with* $r(G) = rad(G) = 1$, $diam(G) = 2$ *with t central vertices. Then* $\gamma_{ed}(G) = 2$ *if and only if any one of the following is true.*

(i) *G has at least one vertex of degree t.*

(ii) *There exist* $u, v \in V(G)$ *such that* $D = \{u, v\}$ *is a maximal independent set and* $d_{\langle E_2 \rangle}(u, v) \geq 3$, *that is* $e(u) = 2 = e(v)$ *and a vertex of eccentricity two is adjacent to exactly one of u and v.*

Theorem 1.8 [19]. *If G is a two self-centered graph, then* $\gamma_{ed}(G) = 2$ *if and only if G has a dominating edge which is not in a triangle.*

Theorem 1.9 [19]. *If G is a graph with radius two and diameter three then* $\gamma_{ed}(G) = 2$ *if and only if G has a γ-set* $D = \{u, v\}$ *of cardinality two with* $d\{u, v\} = 3$ *and for any* $u - v$ *path in* G, $e(u) = e(v) = 3$ *and* $e(x) = e(y) = 2$.

2. Geodetic Eccentric Domination in Graphs

In [20], Nishanthi defined Geodetic eccentric domination number. She did not study any properties or bounds about this domination number. In this paper, we study some bounds for Geodetic eccentric domination number and characterize graphs for which Geodetic eccentric domination number $\gamma_{ged}(G) = 2,\ p-1$ and p.

A set S of vertices in a graph G is a geodetic eccentric dominating set if S is both a geodetic set and an eccentric dominating set. The minimum cardinality of a geodetic eccentric dominating set is the geodetic eccentric domination number of G and is denoted by $\gamma_{ged}(G)$. We have, $\gamma(G) \le \gamma_{ed}(G) \le \gamma_{ged}(G)$. Also, $\gamma_g(G) \le \gamma_{ged}(G)$. $\gamma_{ged}(G)$ exists for all graphs, since $V(G)$ is always a geodetic eccentric dominating set. By Theorem 1.3, $2 \le \gamma_g(G) \le \gamma_{ged}(G)$. Any geodetic eccentric dominating set must contains all pendant vertices of G.

For any graph G, $\gamma_{ged}(G) \le \gamma_g(G) + e(G)$ and $\gamma_{ged}(G) \le \gamma_{ed}(G) + g(G)$, where $g(G)$ is geodetic number of G and $e(G)$ is eccentric number of G.

Example 2.1.

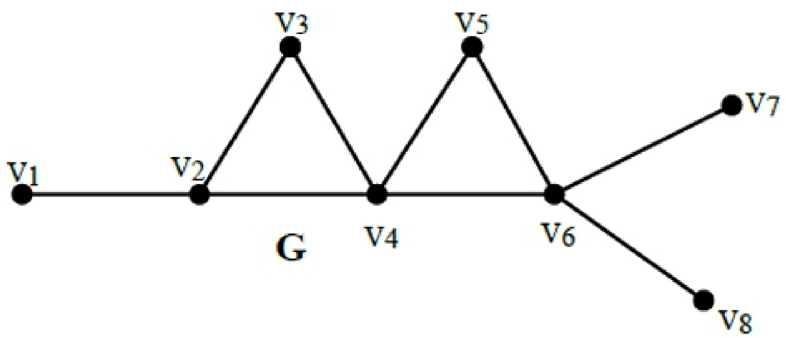

Figure 2.1

In Figure 2.1, $S_1 = \{v_2, v_6\}$ is a minimum dominating set of G, $\gamma(G) = 2$.

$S_2 = \{v_1, v_4, v_7, v_8\}$ is a minimum eccentric dominating set of G, $\gamma_{ed}(G) = 4$.

Example 2.2.

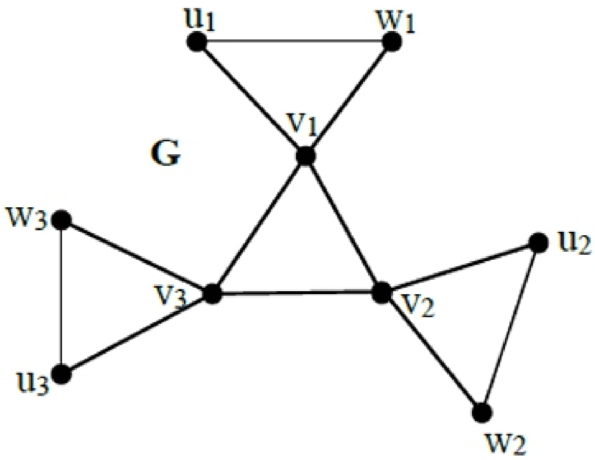

Figure 2.2

In Figure 2.2, $S_1 = \{v_1, v_2, v_3\}$ is a minimum dominating set of G, $\gamma(G) = 3$.

$S_2 = \{u_1, u_2, u_3\}$ is a minimum eccentric dominating set of G, $\gamma_{ed}(G) = 3$.

$S_3 = \{u_1, u_2, u_3, w_1, w_2, w_3\}$ is a minimum geodetic dominating set of G and is a minimum geodetic eccentric dominating set of G, $\gamma_g(G) = \gamma_{ged}(G) = 6$.

In the following theorems, we discuss about the geodetic eccentric dominating set and find out some more bounds for $\gamma_{ged}(G)$.

Theorem 2.1. *Let G be a graph of radius one and diameter two. Then $D \subseteq V(G)$ is a geodetic eccentric dominating set if and only if $D \subseteq E_2$ and is a geodetic eccentric dominating set of the subgraph $\langle E_2 \rangle$.*

Proof. G is a graph with radius one and diameter two. Hence, $\gamma_{ged}(G) \geq 2$. Assume that $D \subseteq E_2$ is a geodetic eccentric dominating set of the subgraph $\langle E_2 \rangle$. This implies that $I[D] = E_2$ and D has at least two vertices of eccentricity two at distance two. Hence, $I[D]$ in G contains all central vertices also. Thus, $D \subseteq E_2$ and D is a geodetic eccentric dominating set of G. Converse is obvious.

Theorem 2.2. *If a connected G has pendant vertices, then any* $\gamma_{ged}(G)$*-set of G contains all its pendant vertices.*

Proof. Every geodetic eccentric dominating set is a geodetic set. Therefore, $\gamma_{ged}(G)$-set of G must contain all pendant vertices of G.

Theorem 2.3. *Let G be a graph of radius one and diameter two and let* $|E_1| = t$ *and* $|E_2| = s$. *Then* $\gamma_{ged}(G) \leq s$.

Proof. E_2 contains at least two non-adjacent vertices. Hence, $I[E_2] = V(G)$. Therefore, E_2 is a geodetic eccentric dominating set of G. Hence, $\gamma_{ged}(G) \leq s$.

Theorem 2.4. *If G is a self-centered graph of diameter two, then* $\gamma_{ged}(G) \leq p - \Delta(G)/2$.

Proof. Let $u \in V(G)$ such that $\deg u = \Delta(G)$. The vertex u and $N_2(u)$ dominate all other vertices of G. Each vertex of $N_2(u)$ is an eccentric vertex of u in G. $N_2(u) \cup \{u\}$ is a geodetic set of G. Hence, $N_2(u) \cup \{u\}$ is a geodetic dominating set of G. Vertices in $N(u)$ may have eccentric vertices in $N(u)$. In this case, let S be a subset of $N(u)$ such that vertices in $N(u) - S$ have eccentric vertices in S. Thus, $|S| \leq \Delta(G)/2$ and $S \cup N_2(u) \cup \{u\}$ is a geodetic eccentric dominating set. Hence, $\gamma_{ged}(G) \leq \Delta(G)/2 + (p - \Delta(G)) = p - \Delta(G)/2$.

Theorem 2.5. *If G is a graph of radius greater than two, then* $\gamma_{ged}(G) \leq p - \delta(G)$.

Proof. Let $u \in V(G)$ such that u is not a support vertex in G. $V(G) - N(u)$ dominate all other vertices of G. Also, since radius > 2, each vertex in $N(u)$ has eccentric vertices in $V - N(v)$ only. $D = V(G) - N(u)$ is a geodetic set, since u is not a support. D is also a geodetic eccentric dominating set. Thus, $\gamma_{ged}(G) \leq |V - N(u)| \leq p - \delta(G)$.

Corollary 2.1. *If there exists u such that* $\deg u = \Delta(G)$ *and u is not a support then* $\gamma_{ged}(G) \leq p - \Delta(G)$.

Theorem 2.6. *If G is a tree, then* $\gamma_{ged}(G) \leq \gamma(G) + t$, *t-number of pendant vertices.*

Proof. Let d be the diameter of G. Let t be the number of pendant vertices of G. Let D be any dominating set of G and S be the set of all pendant vertices of G. Then $D \cup S$ is a geodetic eccentric dominating set of G. Hence, $\gamma_{ged}(G) \leq \gamma(G) + t$.

Theorem 2.7. *If G is a tree of order* $p \geq 3$, *then the following conditions are equivalent.*

(i) $\gamma_{ged}(G) = \gamma_g(G) = \gamma(G) = g(G)$.

(ii) $L(G)$ *is a minimum dominating set of G, where* $L(G)$ *is the number of pendant vertices.*

Proof. $L(G)$ is a minimum geodetic set of a tree G. Also, since eccentric vertices are in $L(G)$, (i) and (ii) are equivalent.

Following theorems characterize graphs for which $\gamma_{ged}(G) = 2,\ p - 1$ and

Theorem 2.8. *Let G be a graph with radius one and diameter two. Then* $\gamma_{ged}(G) = 2$ *if and only if* $G = \overline{K_2} + K_{p-2}$.

Proof. Assume $\gamma_{ged}(G) = 2$. This implies that $\gamma_{ed}(G) = 2$. We know by Theorem 1.7, $\gamma_{ed}(G) = 2$ if and only if G is any one of the following.

(i) G has a vertex of degree m, where m is the number of central vertices.

(ii) G has vertices u, v with $e(u) = e(v) = 2$ such that each vertex of eccentricity two is adjacent to either u or v.

If G has a vertex of degree m then $D = \{x, y\}, e(y) = 1, e(x) = 2$ and $\deg x = m$ is a γ_{ed}-set but in this case D is not geodetic, since $d(x, y) = 1$.

Let $D = \{x, y\}$ be a γ_{ged}-set with $e(x) = e(y) = 2$. Vertices x and y are not adjacent, since D is a geodetic set. If E_2 has more than two vertices by (ii) any vertex of E_2 is not adjacent to both x and y in G. Hence, D is not geodetic if E_2 has more than two vertices.

Hence, $E_2 = \{x, y\} = D$ and $E_1 = V(G) - \{x, y\}$. That is, $G = \overline{K_2} + K_{p-2}$.

Conversely, when $G = \overline{K_2} + K_{p-2}$, it is clear that $\gamma_{ged}(G) = 2$.

Theorem 2.9. *Let G be a two self-centered graph. Then* $\gamma_{ged}(G) \neq 2$.

Proof. Suppose $\gamma_{ged}(G) = 2$. Then $\gamma(G) = \gamma_{ed}(G) = \gamma_{ged}(G) = 2$, since $\gamma_{ed}(G) \neq 1$ and $\gamma(G) \neq 1$ for G. $\gamma_{ed}(G) = 2$ implies that there is a γ_{ed}-set $D = \{x, y\}$ such that $e = xy$ is a dominating edge of G which is not in a triangle by Theorem 1.8. But D is a geodetic eccentric dominating set implies that x and y are not adjacent in G. Hence, $\gamma_{ged}(G)$ cannot be two.

Theorem 2.10. *Let G be a connected graph of radius two and diameter three. Then* $\gamma_{ged}(G) = 2$ *if and only if G has only two peripheral vertices such that all other vertices lie on a diametral path from x to y.*

Proof. We know that $\gamma_{ged}(G) = 2$ implies that $\gamma_{ed}(G) = 2$. But by Theorem 1.9, $\gamma_{ed}(G) = 2$ if and only if G has a γ-set $D = \{x, y\}$ such that

$d(x, y) = 3$, $e(x) = e(y) = 3$ and for any shortest path $xuvy$ in G, $e(u) = e(v) = 2$.

But, D is a geodetic set in G if and only if all other vertices lie on a diametral path from x to y.

Suppose there exists $z \in V(G)$ such that $e(z) = 3$, then z is adjacent to x (or y) and $d(z, y) = 2$ (or $d(z, x) = 2$). Thus, z has no eccentric vertex in D. Hence, except x and y, all other vertices are of eccentricity two. This proves the theorem.

Example 2.3.

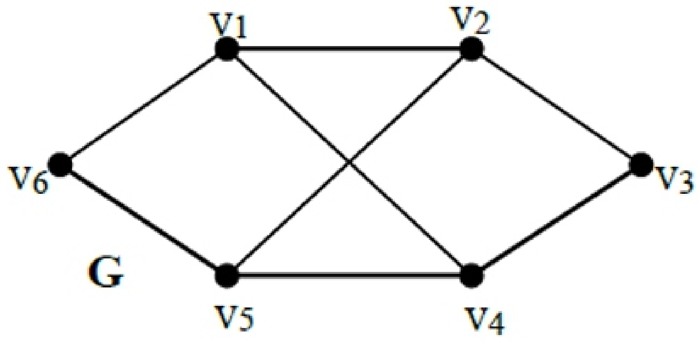

Figure 2.3.

In Figure 2.3, $S = \{v_3, v_6\}$ is a minimum eccentric dominating set of G and is also a minimum geodetic eccentric dominating set of G, $\gamma_{ed}(G) = \gamma_{ged}(G) = 2$.

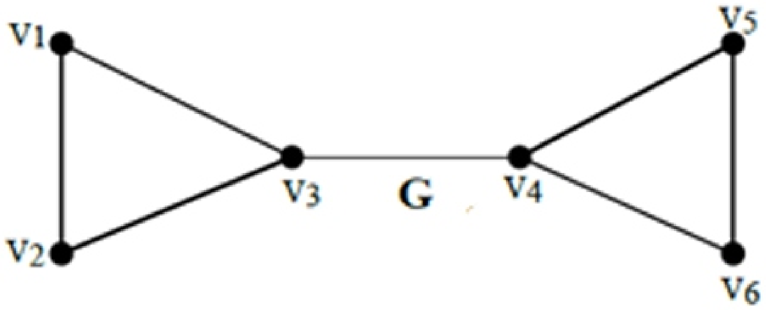

Figure 2.4.

In Figure 2.4, $S_1 = \{v_1, v_5\}$ is a minimum eccentric dominating set of G, $\gamma_{ed}(G) = 2$. $S_2 = \{v_1, v_2, v_5, v_6\}$ is a minimum geodetic eccentric dominating set of G, $\gamma_{ged}(G) = 4$.

Theorem 2.11. *Let G be a graph of radius one and diameter two with $p \geq 4$. Then $\gamma_{ged}(G) = p - 1$ if and only if G is unicentral with $\langle E_2 \rangle$ is disconnected whose components are complete graphs.*

Proof. Let D be γ_{ged}-set with $|D| = p - 1$. Let $v \in V(G)$ such that $D = V - \{v\}$.

Case (i). v is not a central vertex.

Suppose v is not a central vertex. Then D contains a central vertex u and $e(v) = 2$. Vertex u dominates all the vertices, u dominates v and v is eccentric to at least one vertex of E_2. Hence, it is not adjacent to some $w \in E_2$, $w \in D$. Since D is a γ_{ged}-set there exists $x, y \in E_2$ such that xvy is a path P_3 in $\langle E_2 \rangle$. In this case, $D - \{u\}$ is also geodetic eccentric dominating set, which is a contradiction to D is a γ_{ged}-set.

Hence, this case is not possible.

Case (ii). v is a central vertex.

Suppose there exists another central vertex z then $V(G) - \{v, z\}$ is also geodetic eccentric dominating set, which is a contradiction to $\gamma_{ged}(G) = p - 1$. Hence, G is unicentral with centre v and $D = V(G) - \{v\}$ is a geodetic eccentric dominating set. Now, if there exists an induced $P_3 = u_1u_2u_3$ in $\langle E_2 \rangle$, (In this case $p \geq 4$) then $D - \{u_2\}$ is also a geodetic eccentric dominating set, which is a contradiction. Hence, $\langle E_2 \rangle$ has no induced P_3. This implies that $\langle E_2 \rangle$ is disconnected whose components are complete graphs. ($\langle E_2 \rangle$ cannot be complete. If the components are not complete, there exists induced P_3).

Theorem 2.12. *Let G be a self-centered graph of radius two. Then $\gamma_{ged}(G) = p - 1$ if and only if $G = C_4$.*

Proof. Let G be a two self-centered graph. Suppose $\gamma_{ged}(G) = p - 1$, there exists $D = V - \{x\}$ such that $x \in I[D]$. But G is two self-centered implies that G has more than three vertices. Also, $e(x) = 2$ in G implies that $\deg(x) \geq 2$. Since, D is a dominating set and $|D| = p - 1$, there exists $u, v \in D$ such that u and v are adjacent to x. Since G is two self-centered, G is two connected and hence there exists another path from u to v of length two or three.

If $V = \{u, v, x, y\}$. Then x is eccentric to y. Hence $x \notin D$ implies that y must be in D. Hence $G = C_4$, $\gamma_{ged}(G) = 3 = p - 1$. Suppose there exists another path from u to v of length three. Let $uyzv$ be such a path. Then either y or z must be in D. We can get a geodetic eccentric dominating set $V(G) - \{v, z\}$ or $V(G) - \{x, z\}$. Hence, $\gamma_{ged}(G) \neq p - 1$.

Suppose there exists more than two paths from u to v or G has more than four elements. In this case, D is not a minimum geodetic eccentric dominating set. Hence, $\gamma_{ged}(G) = p - 1$ implies that $G = C_4$.

Remark 2.1. If G is a connected graph with radius two and diameter three, then $\gamma_{ged}(G) \neq p - 1$.

Remark 2.2. If G is a connected graph with radius two and diameter four, then $\gamma_{ged}(G) \neq p - 1$.

Remark 2.3. If G is a connected graph with radius greater than two, then $\gamma_{ged}(G) \neq 2$ and $\gamma_{ged}(G) \neq p - 1$.

Theorem 2.13. *Let G be a connected graph. Then $\gamma_{ged}(G) = p - 1$ if and only if* (i) *G is unicentral with centre v and $G - v$ is disconnected whose components are complete graphs.* (ii) $G = C_4$.

Proof. Proof follows from Theorem 2.11 and 2.12 and Remarks 2.1, 2.2, and 2.3.

Theorem 2.14. *Let G be a connected graph. Then $\gamma_{ged}(G) = p$ if and only if $G = K_p$.*

Proof. $\gamma_{ged}(G) = p$ implies that $D = V(G)$ is the only γ_{ged}-set. If there exists non-adjacent vertices then we can find a γ_{ged}-set D such that $|D| < p$.

Hence, any two vertices of G are adjacent to each other. Hence, $G = K_p$ only.

Conversely, suppose $G = K_p$, we know that $\gamma_g(G) = p$. Therefore, $\gamma_{ged}(G) = p$.

3. Geodetic Eccentric Domination in some particular classes of Graphs

The geodetic eccentric domination number of some classes of graphs is given in the following theorems.

Theorem 3.1. *If* K_p *is a complete graph on* p *vertices, then* $\gamma_{ged}(K_p) = p$.

Proof. Let $v_1, v_2, v_3, \ldots, v_p$ be the vertices of the complete graph K_p. $S = \{v_1, v_2, v_3, \ldots, v_p\}$ is the minimum geodetic eccentric dominating set of K_p, where $I[S] = V(K_p)$.

Therefore, $\gamma_{ged}(K_p) = p$.

Theorem 3.2. *If* W_p *is a wheel graph, then*

(i) $\gamma_{ged}(W_p) = \lceil p/2 \rceil$ *if* p *is odd.*

(ii) $\gamma_{ged}(W_p) = p/2$ *if* p *is even,* $p > 4$.

Proof. Let $v, v_1, v_2, v_3, \ldots, v_p$ be the vertices of the wheel graph W_p, where v is the central vertex.

Case (i). p is odd.

$S = \{v_1, v_3, v_5, \ldots, v_{p-2}, v_p\}$ is the minimum geodetic eccentric dominating set of W_p. Hence, $\gamma_{ged}(W_p) = \lceil p/2 \rceil$.

Case (ii). p is even.

$S = \{v_1, v_3, v_5, \ldots, v_{p-3}, v_{p-1}\}$ is the minimum geodetic eccentric dominating set of W_p. Therefore, $\gamma_{ged}(W_p) = p/2$.

Remark 3.1. $\gamma_{ged}(W_4) = 3$.

Theorem 3.3. *If* F_p *is a fan graph, then*

(i) $\gamma_{ged}(F_p) = \lceil p/2 \rceil$ *if p is odd.*

(ii) $\gamma_{ged}(F_p) = p/2 + 1$ *if p is even.*

Proof. Let $w, w_1, w_2, w_3, \ldots, w_p$ be the vertices of the fan graph F_p.

Case (i). p is odd.

$S = \{w_1, w_3, w_5, \ldots, w_{p-2}, w_p\}$ is the minimum geodetic eccentric dominating set of F_p. Hence, $\gamma_{ged}(F_p) = \lceil p/2 \rceil$.

Case (ii). p is even.

$S = \{w_1, w_3, w_5, \ldots, w_{p-1}, w_p\}$ is the minimum geodetic eccentric dominating set of F_p. Therefore, $\gamma_{ged}(F_p) = p/2 + 1$.

Theorem 3.4. *If* $K_{m,n}$ *is a complete bi-partite graph with* $m, n > 2$, *then* $\gamma_{ged}(K_{m,n}) = 4$, *where* $m + n = p$.

Proof. Let $A = \{v_1, v_2, v_3, \ldots, v_m\}$ and $B = \{w_1, w_2, w_3, \ldots, w_n\}$ be the

set of vertices of $K_{m,n}$. $S = \{v_1, v_2, w_1, w_2\}$ is the minimum geodetic eccentric dominating set of $K_{m,n}$.

Therefore, $\gamma_{ged}(K_{m,n}) = 4$.

Remark 3.2. $\gamma_{ged}(K_{1,n}) = n, n > 1$ and $\gamma_{ged}(K_{2,n}) = 3, n > 1$.

Theorem 3.5. *If* $K_{1,n}$ *is a star graph, then* $\gamma_{ged}(K_{1,n}) = n, n > 1$, *where* $p = n + 1$.

Proof. Let $v, v_1, v_2, v_3, \ldots, v_n$ be the vertices of the star graph $K_{1,n}$, where v is the central vertex of $K_{1,n}$. $S = \{v_1, v_2, v_3, \ldots, v_n\}$ is the minimum geodetic eccentric dominating set of $K_{1,n}$, where $I[S] = V(K_{1,n})$. Therefore, $\gamma_{ged}(K_{1,n}) = n, n > 1$, where $p = n + 1$.

Remark 3.3. $\gamma_{ged}(K_{1,n}) = 2.$

Theorem 3.6. *If* P_p *is a path graph,* $p \geq 3$. *Then*

(i) $\gamma_{ged}(P_p) = \lceil p/3 \rceil$ *if* $p = 3k + 1$.

(ii) $\gamma_{ged}(P_p) = \lceil p/3 \rceil + 1$ *if* $p = 3k$ *or* $3k + 2$.

Proof. Let $v_1, v_2, v_3, \ldots, v_p$ represent the path P_p.

Case (i). $p = 3k$.

$S = \{v_2, v_5, v_8, \ldots, v_{3k-1}\}$ is the only minimum dominating set in P_p.

$S' = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k}\}$ is a geodetic dominating set of P_p. S' is also a geodetic eccentric dominating set of P_p.

Thus, $\gamma_{ged}(P_p) = \gamma_g(P_p) = \lceil p/3 \rceil + 1$.

Case (ii). $p = 3k + 1$.

$S = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k+1}\}$ is the minimum geodetic dominating set of P_p. S is also a geodetic eccentric dominating set of P_p. Thus, $\gamma_{ged}(P_p) = \gamma_g(P_p) = \lceil p/3 \rceil$.

Case (iii). $p = 3k + 2$.

$S = \{v_1, v_2, v_5, v_8, \ldots, v_{3k+2}\}$ is the minimum geodetic dominating set of P_p. S is also a geodetic eccentric dominating set of P_p. Thus, $\gamma_{ged}(P_p) = \gamma_g(P_p) = \lceil p/3 \rceil + 1$.

Theorem 3.7. *If* C_p *is a cycle graph,* $p \geq 6$, *then*

(i) $\gamma_{ged}(C_p) = p/2$ *if p is even.*

(ii) $\gamma_{ged}(C_p) = \lceil p/3 \rceil$ *(or)* $\lceil p/3 \rceil + 1$ *if p is odd.*

Proof of (i). Let $p = 2k$ and $k > 2$.

Let the cycle C_p be $v_1 v_2 v_3 \ldots v_{2k} v_1$. Each vertex of C_p has exactly one

eccentric vertex (that is C_p unique eccentric vertex graph).

Hence, $\gamma_{ged}(C_p) \geq \gamma_g(C_p) \geq \gamma_{ed}(C_p) \geq p/2$. (1)

Case (i). k-odd.

$S = \{v_1, v_3, \ldots, v_k, v_{k+2}, \ldots, v_{2k-1}\}$ is the minimum geodetic dominating set of C_p. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq p/2$. (2)

From (1) and (2), $\gamma_{ged}(C_p) = p/2$.

Case (ii). k-even.

$S = \{v_1, v_3, \ldots, v_{k-1}, v_{k+2}, \ldots, v_{2k}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertices of v_{i+k}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq p/2$. (3)

From (1) and (3), $\gamma_{ged}(C_p) = p/2$.

Proof of (ii). When p is odd, each vertex of C_p has exactly two eccentric vertices. If $p = 2k+1$, $v_i \in V(G)$ has v_{i+1}, v_{i+k+1} as eccentric vertices.

Case (i). $p = 3m$, $m \geq 3$.

Also $p = 3m$, p is odd $\Rightarrow m$ is odd.

$S = \{v_1, v_4, \ldots, v_k, v_{k+3}, \ldots, v_{2k-1}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertex of v_{i+k} and v_{i+k+1}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq \lceil p/3 \rceil$. (4)

By Theorem 1.6, $\gamma_{ged}(C_p) = \lceil p/3 \rceil \leq \gamma_{ged}(C_p)$. (5)

From (4) and (5), $\gamma_{ged}(C_p) = \lceil p/3 \rceil$.

Case (ii). $p = 3m+1$, $m \geq 2$.

Also $p = 3m + 1$, p is odd $\Rightarrow m$ is even.

$S = \{v_1, v_4, \ldots, v_{k+1}, v_{k+3}, v_{k+6}, \ldots, v_{2k-1}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertex of v_{i+k} and v_{i+k+1}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq \lceil p/3 \rceil$. (6)

By Theorem 1.6, $\gamma_{ed}(C_p) = \lceil p/3 \rceil \leq \gamma_{ged}(C_p)$. (7)

From (6) and (7), $\gamma_{ged}(C_p) = \lceil p/3 \rceil$.

Case (iii). $p = 3m + 2$, $m \geq 1$.

Also $p = 3m + 2$, p is odd $\Rightarrow m$ is odd.

$S = \{v_1, v_4, \ldots, v_{k-1}, v_k, v_{k+3}, \ldots, v_{2k+1}\}$ is the minimum geodetic dominating set of C_p. Vertex V_i is an eccentric vertex of v_{i+k} and v_{i+k+1}. S is also a geodetic eccentric dominating set of C_p. Therefore, $\gamma_{ged}(C_p) \leq \lceil p/3 \rceil + 1$. (8)

By Theorem 1.6, $\gamma_{ed}(C_p) = \lceil p/3 \rceil + 1 \leq \gamma_{ged}(C_p)$. (9)

From (8) and (9), $\gamma_{ged}(C_p) = \lceil p/3 \rceil + 1$.

Remark 3.4. $\gamma_{ged}(C_3) = \gamma_{ged}(C_4) = \gamma_{ged}(C_5) = 3$.

Theorem 3.8. *If $P_n \circ K_1$ is a path corona, then $\gamma_{ged}(P_n \circ K_1) = n$, where $2n = p$.*

Proof. Let $A = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of P_n and $B = \{w_1, w_2, w_3, \ldots, w_n\}$ be the set of pendant vertices attached at $v_1, v_2, v_3, \ldots, v_n$ respectively. $S = \{w_1, w_2, w_3, \ldots, w_n\}$ is the minimum eccentric dominating set of $P_n \circ K_1$. $I[S] = V(P_n \circ K_1)$. Hence, S is also a geodetic eccentric dominating set. Therefore, $\gamma_{ged}(P_n \circ K_1) = n$, where $2n = p$.

Theorem 3.9. *If $C_n \circ K_1$ is a cycle corona, then $\gamma_{ged}(C_n \circ K_1) = n$, where $2n = p$.*

Proof. Let $A = \{v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of C_n and $B = \{w_1, w_2, w_3, \ldots, w_n\}$ be the set of pendant vertices attached at $v_1, v_2, v_3, \ldots, v_n$ respectively. $S = \{w_1, w_2, w_3, \ldots, w_n\}$ is the minimum eccentric dominating set of $C_n \circ K_1$. $I[S] = V(C_n \circ K_1)$. Hence, S is also a geodetic eccentric dominating set. Therefore, $\gamma_{ged}(C_n \circ K_1) = n$, where $2n = p$.

Theorem 3.10. *If $K_{1,n} \circ K_1$ is a star corona, then $\gamma_{ged}(K_{1,n} \circ K_1) = n + 1$, where $2n + 2 = p$.*

Proof. Let $A = \{v, v_1, v_2, v_3, \ldots, v_n\}$ be the set of vertices of $K_{1,n}$ and $B = \{w, w_1, w_2, w_3, \ldots, w_n\}$ be the set of pendant vertices attached at $v, v_1, v_2, v_3, \ldots, v_n$ respectively. $S = \{w, w_1, w_2, w_3, \ldots, w_n\}$ is the minimum eccentric dominating set of $K_{1,n} \circ K_1$. $I[S] = V(K_{1,n} \circ K_1)$. Hence, S is also a geodetic eccentric dominating set.

Therefore, $\gamma_{ged}(K_{1,n} \circ K_1) = n + 1$, where $2n + 2 = p$.

Theorem 3.11. *If $K_{1,n,n}$ is a spider, then $\gamma_{ged}(K_{1,n,n}) = n + 1$ where*

$2n + 1 = p$.

Proof. Let $K_{1,n,n}$ be a spider. Let w be a vertex of maximum degree $\Delta(K_{1,n}, n)$ and S be the set of pendant vertices. The set $S \cup \{w\}$ form a minimum geodetic eccentric dominating set, where $I[S \cup \{w\}] = V(K_{1,n}, n)$. Therefore, $\gamma_{ged}(K_{1,n}, n) = n + 1$.

Theorem 3.12. *If G is a wounded spider (not a path), then $\gamma_{ged}(G) = r + h$, where r is the number of non-wounded legs, h is the number of wounded legs.*

Proof. Let G be a wounded spider. Let w be a vertex of maximum degree $\Delta(G)$ and R be the set of pendant vertices which are adjacent to vertices of degree two, H be the set of pendant vertices which are adjacent to w. The set $R \cup H$ form a minimum geodetic eccentric dominating set, where $I[R \cup H] = V(G)$, $|R| = r$, $|H| = h$. Therefore, $\gamma_{ged}(G) = r + h$.

Conclusion

Here, we have studied geodetic eccentric domination in some families of graphs and also found out some bounds for geodetic eccentric domination number of a graph. Also, we have characterized graphs for which $\gamma_{ged}(G) = 2,\ p-1,$ and p.

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FIXED POINT THEOREMS FOR MAPPINGS INVOLVING RATIONAL TYPE EXPRESSIONS IN DUALISTIC PARTIAL METRIC SPACES

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ABSTRACT

The aim of this paper is to establish some fixed point theorems for mappings involving rational expressions in a complete dualistic partial metric space using a class of pairs of functions satisfying certain assumptions. Our result extends and generalizes some well-known results of [8], [9], [26] and [33]. We also provide examples which show the usefulness of these results.

Keywords: fixed point, dualistic partial metric, dualistic contractions.

1. Introduction

Matthews [17] introduced a new generalized metric space called partial metric space. He established the precise relationship between partial metric spaces and the so-called weightable quasi-metric spaces. After this contribution, many researchers focused on partial metric spaces (see [1], [11], [12], [13], [14], [15], [22], [28]).

The concept of dualistic partial metric, which is more general than partial metric, was studied by O'Neill [29] and established a robust relationship between dualistic partial metric and quasi metric. For the more details of fixed point results on dualistic partial metric spaces, the readers may refer to [4], [16], [19] [20], [23], [25], [27], [28].

Das and Gupta [8] established first fixed point theorem for rational contractive type conditions in metric space.

Theorem 1.1 (see [8]). *Let (X, d) be a complete metric space, and let $T : X \to X$ be a self-mapping. If there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that*

$$d(Tx, Ty) \le \alpha d(x, y) + \beta \frac{[1 + d(x, Tx)]d(y, Ty)}{1 + d(x, y)} \tag{1.1}$$

for all $x, y \in X$, then T has a unique fixed point $x^* \in X$.

Nazam et al. [26] proved a real generalization of Das-Gupta fixed point theorem in the frame work of dualistic partial metric spaces. The main purpose of this paper is to present some fixed point theorems for mappings involving rational expressions in the context of complete dualistic partial metric spaces using a class of pairs of functions satisfying certain assumptions. Our result extends and generalizes some well-known results of [8], [9], [26] and [33]. We also provide examples to show significance of the obtained results involving rational type dualistic contractive conditions.

2. Preliminaries

We recall some mathematical basics and definitions to make this paper self-sufficient.

Definition 2.1 (see [17]). Let X be a non-empty set. A partial metric on X is a function $p : X \times X \to [0, \infty)$ complying with following axioms, for all $x, y, z \in X$

$(p_1) x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y);$

$(p_2) p(x, x) \leq p(x, y);$

$(p_3) p(x, y) = p(y, x);$

$(p_4) p(x, y) \leq p(x, z) + p(v, y) - p(z, z).$

The pair (X, p) is called a partial metric space.

Definition 2.2 (see [29]). Let X be a non-empty set. A dualistic partial metric on X is a function $p^* : X \times X \to (-\infty, \infty)$ satisfying the following axioms, for all $x, y, z \in X$

$(p_1^*) x = y \Leftrightarrow p^*(x, y) = p^*(x, x) = p^*(y, y);$

$(p_2^*) p^*(x, x) \leq p^*(x, y);$

$(p_3^*) p^*(x, y) = p^*(y, x);$

$(p_4^*) p^*(x, z) + p^*(y, y) \leq p^*(x, y) + p^*(y, z).$

The pair (X, p^*) is called a dualistic partial metric space.

Remark 2.3. Noting that each partial metric is a dualistic partial metric but the converse is false. Indeed, define a function p^* on $(-\infty, \infty)$ as $p^*(x, y) = \max\{x, y\}, \forall x, y \in (-\infty, \infty)$. Obviously, p^* is a dualistic partial metric on $(-\infty, \infty)$. Since $p^*(x, y) < 0 \notin [0, \infty), \forall x, y \in (-\infty, 0)$ and then p^* is not a partial metric on $(-\infty, \infty)$. This confirms our remark. Unlike other metrics, in dualistic partial metric $p^*(x, y) = 0$ does not imply $x = y$. Indeed, for all $k > 0$, $p^*(-k, 0) = 0$ and $-k \neq 0$. The self-distance $p^*(x, x)$ is a feature utilized to describe the amount of information contained in x. The restriction of p^* to $[0, \infty)$ is a partial metric. This situation creates a problem in obtaining a fixed point of a self-mapping in dualistic partial metric space. For the solution of this problem, Nazam et al. [21] introduced concept of convergence comparison property (CCP) and established some fixed point by using (CCP) along with axioms (p_1^*) and (p_2^*).

Definition 2.4 (see [21]). Let (X, p^*) be a dualistic partial metric space and $\mathcal{T}$ be a self-mapping on X. We say that $\mathcal{T}$ has a convergence comparison property (CCP) if for each sequence $\{x_n\}$ in X such that $x_n \to x$, $\mathcal{T}$ satisfies

$$p^*(x, x) \leq p^*(\mathcal{T}x, \mathcal{T}x) \tag{2.1}$$

Example 2.5. Let $X = (-\infty, 0]$, $X_1 = (-4, 0]$, $X_2 = (-\infty, -4]$. Define a mapping $p^* : X \times X \to (-\infty, \infty)$ by $p^*(x, y) = |x - y|$ if $x \neq y$ and $p^*(x, y) = x \vee y$ if $x = y$. Clearly, (X, p^*) is a complete dualistic partial metric space. Consider $\left\{x_n = \frac{1}{n} - 2, n \geq 1\right\}_{n \in \mathbb{N}} \subset X$. Here $\lim_{n\to\infty} p^*(x_n, -2) = p^*(-2, -2) \Rightarrow \lim_{n\to\infty} x_n = -2$ in (X, p^*). Define $\mathcal{T} : X \to X$ by $\mathcal{T}x = -1$ if $x \in X_2$ and $\mathcal{T}x = 0$ if $x \in X_1$. For such $x = -2$, observe that $p^*(x, x) = x \vee x = x = -2 \leq 0 = p^*(0, 0) = p^*(\mathcal{T}(-2), \mathcal{T}(-2))$. So $\mathcal{T}$ has the (CCP).

Example 2.6 (see [21], [29]). (1) Define $p_d^* : X \times X \to (-\infty, \infty)$ by $p_d^*(x, y) = d(x, y) + b$, where d is a metric on a nonempty set X and $b \in (-\infty, \infty)$ is arbitrary constant, then it is easy to check that p_d^* verifies axioms $(p_1^*) - (p_4^*)$ and hence (X, p^*) is a dualistic partial metric space.

(2) Let p be a partial metric defined on a non empty set X. The function $p^* : X \times X \to (-\infty, \infty)$ defined by $p^*(x, y) = p(x, y) - p(x, x) - p(y, y)$ satisfies the axioms $(p_1^*) - (p_4^*)$ and so it defines a dualistic partial metric on X. Note that $p^*(x, y)$ may have negative values.

(3) Let $X = (-\infty, \infty)$. Define $p^* : X \times X \to (-\infty, \infty)$ by $p^*(x, y) = | x - y |$ if $x \neq y$ and $p^*(x, y) = -\beta$ if $x = y$ and $\beta > 0$. We can easily see that p^* is a dualistic partial metric on X.

O'Neill [29] established that each dualistic partial metric p^* on X generates a T_0 topology $\tau(p^*)$ on X having a base, the family of p^*- balls

$\{\mathcal{B}_{p^*}(x, \epsilon) | x \in X, \epsilon > 0\}$, where

$$\{\mathcal{B}_{p^*}(x, \epsilon) = \{y \in X | p^*(x, y) < p^*(x, x) + \epsilon\}.$$

If (X, p^*) is a dualistic partial metric space, then the function $d_{p^*} : X \times X \to [0, \infty)$ defined by

$$d_{p^*}(x, y) = p^*(x, y) - p^*(x, x) \tag{2.2}$$

defines a quasi-metric on X such that $\tau(p^*) = \tau(dp^*)$ and

$$d^s_{p^*}(x, y) = \max\{d_{p^*}(x, y), d_{p^*}(y, x)\} \tag{2.3}$$

defines a metric on X.

Definition 2.7 (see [28]). Let (X, p^*) be a dualistic partial metric space.

1. A sequence $\{x_n\}$ in X is said to converge or to be convergent if there is

a $x \in X$ such that $\lim_{n\to\infty} p^*(x_n, x) = p^*(x, x)$. x is called the limit of $\{x_n\}$ and we write $x_n \to x$.

2. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if $\lim_{n,m\to\infty} p^*(x_n, x_m)$ exists and is finite.

3. A dualistic partial metric space $X = (X, p^*)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau(p^*)$, to a point $x \in X$ such that $p^*(x, x) = \lim_{n,m\to\infty} p^*(x_n, x_m)$.

Remark 2.8. For a sequence, convergence with respect to metric space may not imply convergence with respect to dualistic partial metric space. Indeed, if we take $\beta = 1$ and $\{x_n = \frac{1-n}{n} : n \geq 1\}_{n\in\mathbb{N}} \subset X$ as in Example 2.6 (3). Mention that $\lim_{n\to\infty} d(x_n, -1) = -1$ and therefore, $x_n \to -1$ with respect to d. On the other hand, we make a conclusion that $x_n \not\to -1$ with respect to p^* because $\lim_{n\to\infty} p^*(x_n, -1) = \lim_{n\to\infty} p^*|x_n - (-1)|$ $= \lim_{n\to\infty}\left|\frac{1-n}{n} + 1\right| = 0$ and $p^*(-1, -1) = -1$.

Lemma 2.9 (see [28]). *Let (X, p^*) be a dualistic partial metric space.*

(1) *Every Cauchy sequence in $(X, d^s_{p^*})$ is also a Cauchy sequence in (X, p^*).*

(2) *A dualistic partial metric (X, p^*) is complete if and only if the induced metric space $(X, d^s_{p^*})$ is complete.*

(3) *A sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to $\tau(d^s_{p^*})$ if and only if $p^*(x, x) = \lim_{n\to\infty} p^*(x_n, x) = \lim_{n\to\infty} p^*(x_n, x_m)$.*

Definition 2.9 (see [26]). Let (X, p^*) be a dualistic partial metric space. A mapping $\mathcal{T} : X \to X$ is said to be a dualistic Dass-Gupta contraction if there exist $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$ such that

Nazam et al. [26] studied the following fixed point theorems on dualistic contraction of rational type.

Theorem 2.10. *Let* (X, p^*) *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a dualistic Das-Gupta contraction. If* $\mathcal{T}$ *satisfies* (CCP)*. Then* $\mathcal{T}$ *has a unique fixed point in* X *and the Picard iterative sequence* $\{\mathcal{T}_n(x_0)\}$ *with initial point* x_0*, converges to the fixed point.*

One of the most important ingredients of a contractive condition is to study the kind of involved functions, like altering distance functions introduced by Khan et al. [16] as follows.

Definition 2.11 (see [16])**.** A function $\varphi: [0, \infty) \to [0, \infty)$ is said to be altering distance function if

(a1) φ is monotone increasing and continuous,

(a2) $\varphi(t) = 0 \Leftrightarrow t = 0, \forall t \in [0, \infty)$.

Definition 2.12 (see [5])**.** The pair (φ, ϕ), where $\varphi, \phi : [0, \infty) \to [0, \infty)$ is called a pair of generalized altering distance functions if

(b1) φ is continuous;

(b2) φ is non-decreasing;

(b3) $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$.

The condition (b3) was introduced by Moradi and Farajzadeh [18]. The above conditions do not determine the values of $\varphi(0)$ and $\phi(0)$.

Definition 2.13 (see [2])**.** We will denote by $\mathcal{F}$ the family of all pairs (φ, ϕ), where $\varphi, \phi : [0, \infty) \to [0, \infty)$ are functions satisfying the following conditions.

(F1) φ is non-decreasing;

(F2) if $\exists t_0 \in [0, \infty)$ such that $\phi(t_0) = 0$, then $t_0 = 0$ and $\varphi^{-1}(0) = \{0\}$.

(F3). if $\{\alpha_n\}, \{\beta_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lambda$ satisfying $\lambda < \{\beta_n\}$ and $\varphi(\beta_n) \leq (\varphi - \phi)(\alpha_n), \forall n \in \mathbb{N}$, then $\lambda = 0$.

Definition 2.14 (see [33]). A pair of functions (φ, ϕ) is said to belong to the class $\mathfrak{F}$ if they satisfy the following conditions:

(c1) $\varphi, \phi : [0, \infty) \to [0, \infty)$;

(c2) if $t, s \in [0, \infty), \varphi(t) \leq \phi(s)$ then $t \leq s$;

(c3). if $\{t_n\}, \{s_n\} \subset [0, \infty), \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta$ and $\varphi(t_n) \leq \phi(s_n), \forall n \in \mathbb{N}$, then $\delta = 0$.

If (φ, ϕ) satisfies (F1) and (F2), then $(\varphi, \phi = \varphi - \phi)$ satisfies (c1) and (c2). Furthermore, if $(\varphi, \phi = \varphi - \phi)$ satisfies (c3), then (φ, ϕ) satisfies (F3).

Remark 2.15 (see [33]). If $(\varphi, \phi) \in \mathfrak{F}$ and $\varphi(t) \leq \phi(t)$, then $t = 0$, since we can take $t_n = s_n, t, \forall n \in \mathbb{N}$ and by (c3), we deduce $t = 0$.

Example 2.16. The conditions (c1)-(c3) of Definition 2.14 are fulfilled for the functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ defined by

(1) $\varphi(t) = \ln\left(\frac{5t+1}{12}\right)$ and $\phi(t) = \ln\left(\frac{3t+1}{12}\right), \forall t \in [0, \infty)$.

(2) $\varphi(t) = \ln\left(\frac{2t+1}{2}\right)$ and $\phi(t) = \ln\left(\frac{t+1}{2}\right), \forall t \in [0, \infty)$.

Example 2.17 (see [33]). Let $\mathcal{S} = \{\ell : [0, \infty) \to [0, \infty) | \ell(t_n) \to 1 \Rightarrow t_n \to 0\}$. Consider the pairs of functions $(1_{[0, \infty)}, \ell(1_{[0, \infty)}))$, where $\ell \in \mathcal{S}$ and $\ell(1_{[0, \infty)})$ is defined as

$$(\ell(1_{[0, \infty)}))(t) = \ell(t)t, \forall t \in [0, \infty).$$

It is easy to check that $(1_{[0, \infty)}, \ell(1_{[0, \infty)})) \in \mathfrak{F}$.

Example 2.18 (see [33]). Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function such that $\varphi(t) = 0 \Leftrightarrow t = 0, \forall t \in [0, \infty)$. Let

$\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\phi(t) = 0 \Leftrightarrow t = 0, \forall t \in [0, \infty)$ and $\phi \leq \varphi$. We make a conclusion that $(\varphi, \varphi - \phi) \in \mathfrak{F}$.

An interesting particular case is when φ is the identity mapping, $\varphi = 1_{[0, \infty)}$ and $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function such that $\phi(t) = 0 \Leftrightarrow t = 0$ and $\phi(t) \leq t, \forall t \in [0, \infty)$.

Remark 2.19 (see [33]). Let $g : [0, \infty) \to [0, \infty)$ be an increasing function and $(\varphi, \phi) \in \mathfrak{F}$. Then $(g \circ \varphi, g \circ \phi) \in \mathfrak{F}$.

3. Main Results

In this section, using the class $\mathfrak{F}$ functions, we give generalizations of some fixed point theorems from the literature.

Theorem 3.1. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \times X$ be a mapping such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying*

$$\varphi(|\, p^*(\mathcal{T}x, \mathcal{T}y) \,|) \leq \max\{\phi(|\, p^*(x, y) \,|), \phi(|\, \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \,|)\} \quad (3.1)$$

$\forall x, y \in \Delta$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{T_n(x_0)\}$ with initial point x_0, converges to the fixed point.

Proof. Let $x_0 \in X$ be an initial element and define Picard iterative sequence $\{x_n\}$ by $\mathcal{T}x_{n-1} = x_n, \forall n \in \mathbb{N}$. If there is a positive integer n_0 such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. So x_{n_0} is a fixed point of $\mathcal{T}$. In this case, the proof is finished. Now, we suppose that $x_n \neq x_{n+1}, \forall n \in \mathbb{N}$, applying (3.1), we have

$$\varphi(|\, p^*(x_{n+1}, x_n) \,|) = \varphi(|\, p^*(\mathcal{T}x_n, \mathcal{T}x_{n-1}) \,|)$$

$$\leq \max\{\varphi(| p^*(x_n, x_{n-1}) |), \varphi(| \frac{p^*(x_{n-1}, \mathcal{T}x_{n-1})(1 + p^*(x_n, \mathcal{T}x_n))}{1 + p^*(x_n, x_{n-1})} |)\}$$

$$= \max\{\phi(| p^*(x_n, x_{n-1}) |), \phi(| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_n, x_{n-1})} |)\}. \quad (3.2)$$

Now, we can distinguish two cases.

Case 1. Consider

$$\max\{\phi(| p^*(x_n, x_{n-1}) |), \phi(| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_n, x_{n-1})} |)\}$$

$$= \phi(| p^*(x_n, x_{n-1}) |). \quad (3.3)$$

Due to inequality (3.2), we have

$$\phi(| p^*(x_{n+1}, x_n) |) \leq \phi(| p^*(x_n, x_{n-1}) |). \quad (3.4)$$

Since $(\varphi, \phi) \in \mathfrak{F}$, we deduce that

$$| p^*(x_{n+1}, x_n) | \preceq | p^*(x_n, x_{n-1}) |.$$

Case 2. If

$$\max\{\phi(| p^*(x_n, x_{n-1}) |), \phi(| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_n, x_{n-1})} |)\}$$

$$= \phi(| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_{n-1}, x_n)} |) \quad (3.5)$$

Then from (3.3), we have

$$\varphi(| p^*(x_{n+1}, x_n) |) \leq \phi(| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_{n-1}, x_n)} |). \quad (3.6)$$

Since $(\varphi, \phi) \in \mathfrak{F}$ we get

$$| p^*(x_n, x_{n+1}) | \leq | \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_{n-1}, x_n)} |$$

which implies that

$$| p^*(x_{n+1}, x_n) | \leq | p^*(x_n, x_{n-1}) |.$$

From both cases, we conclude that the sequence $\{| p^*(x_{n+1}, x_n) |\}$ is a monotone and bounded below sequence of non-negative real numbers, it is convergent and converges to a point r, i.e. $\lim_{n\to\infty} | p^*(x_{n+1}, x_n) | = r \geq 0$. If $r = 0$. Then we have done. Let $r > 0$ and denote $A = \{n \in \mathbb{N} | n$ satisfies (3.3)$\}$ and $B = \{n \in \mathbb{N} | n$ satisfies (3.5)$\}$. Now, we make the following remark.

(1) If Card $A = \infty$, then from (3.2), we can find infinitely natural numbers n satisfying inequality (3.4) and since $\lim_{n\to\infty} | p^*(x_{n+1}, x_n) | = \lim_{n\to\infty} | p^*(x_n, x_{n-1}) | = r$ and $(\varphi, \phi) \in \mathfrak{F}$, we deduce that $r = 0$.

(2) If Card $B = \infty$, then from (3.2), we can find infinitely many $n \in \mathbb{N}$ satisfying inequality (3.6). Since $(\varphi, \phi) \in \mathfrak{F}$ and using the similar argument

to the one used in case 2, we obtain

$$| p^*(x_n, x_{n+1}) | \leq \left| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_{n-1}, x_n)} \right|$$

(3) for infinitely many $n \in \mathbb{N}$. On letting the limit as $n \to \infty$ and taking into account that $\lim_{n\to\infty} | p^*(x_{n+1}, x_n) | = r$, we deduce that $r \leq \frac{r(1+r)}{1+r}$ and consequently, we obtain $r = 0$.

Therefore, in both cases we have

$$\lim_{n\to\infty} | p^*(x_{n+1}, x_n) | = 0 \text{ and then } \lim_{n\to\infty} | p^*(x_{n+1}, x_n) | = 0. \tag{3.7}$$

We use (3.1) to find the self-distance $p^*(x_n, x_{n-1})$, as follows:

$$\varphi(| p^*(x_n, x_n) |) = \varphi(| p^*(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-1}) |)$$

$$\leq \max\{\varphi(| p^*(x_{n-1}, x_{n-1}) |),$$

$$\varphi(\| \frac{p^*(x_{n-1}, \mathcal{T}x_{n-1})(1 + p^*(x_{n-1}, \mathcal{T}x_{n-1}))}{1 + p^*(x_{n-1}, x_{n-1})} \|)\}$$

$$= \max\{\phi(\| p^*(x_{n-1}, x_{n-1}) \|),$$

$$\phi(\| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} \|)\}. \tag{3.8}$$

Put

$$C = \{n \in \mathbb{N} \,|\, \varphi(\| p^*(x_n, x_n) \|) \leq \phi(\| p^*(x_{n-1}, x_{n-1}) \|)\}$$

$$D = \{n \in \mathbb{N} \,|\, \varphi(\| p^*(x_n, x_n) \|) \leq \phi(\| \frac{p^*(x_{n-1}, x_{n-1})(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} \|)\}.$$

By (3.8), we have Card $C = \infty$ or Card $D = \infty$. If Card $C = \infty$, then there exists infinitely many $n \in \mathbb{N}$ satisfying

$$\varphi(\| p^*(x_n, x_n) \|) \leq \phi(\| p^*(x_{n-1}, x_{n-1}) \|) \tag{3.9}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, we have

$$| p^*(x_n, x_n) | \leq | p^*(x_{n-1}, x_{n-1}) |.$$

Thus, $\{\| p^*(x_{n+1}, x_n) \|\}$ is a non increasing sequence of positive real numbers and arguing like case of (3.7), we have $\lim_{n\to\infty} | p^*(x_n, x_n) | = 0$. On the other hand, if Card $D = \infty$, then we can find infinitely many $n \in \mathbb{N}$ satisfying

$$\varphi(\| p^*(x_n, x_n) \|) \leq \phi(\| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} \|) \tag{3.10}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, we infer

$$| p^*(x_n, x_n) | \leq | \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} | \tag{3.11}$$

taking the $\lim_{n\to\infty}$ on (3.11) and using (3.7), we obtain that

$\lim_{n\to\infty}|\ p^*(x_n,\ x_n)\ | \le 0$ and then $\lim_{n\to\infty}|\ p^*(x_n,\ x_n)\ | = 0$. Thus, in both cases, we infer that $\lim_{n\to\infty}|\ p^*(x_n,\ x_n)\ | = 0$ and then

$$\lim_{n\to\infty} p^*(x_n,\ x_n) = 0. \tag{3.12}$$

We deduce from (2.2) that

$$d_{p^*}(x_n,\ x_{n+1}) = p^*(x_n,\ x_{n+1}) - p^*(x_n,\ x_n).$$

So using (3.7) and (3.12), we get

$$\lim_{n\to\infty} d_{p^*}(x_n,\ x_{n+1}) = 0. \tag{3.13}$$

Next step is to show that $\{x_n\}$ is a Cauchy sequence in $(X,\ d^s_{p^*})$. For this we have to show that

$$\lim_{m,\ n\to\infty} d^s_{p^*}(x_n,\ x_m) = \lim_{m,\ n\to\infty} \max\{d_{p^*}(x_n,\ x_m),\ d_{p^*}(x_m,\ x_n)\} = 0.$$

Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence, that is $\lim_{n,m\to\infty} d_{p^*}(x_n,\ x_m) \ne 0$. Then given $\epsilon > 0$, we will construct a pair of subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is smallest index for which for all $n_k > m_k > k$, where $k \in \mathbb{N}$

$$d_{p^*}(x_{n_k},\ x_{m_k}) \ge \epsilon. \tag{3.14}$$

It follows directly that

$$d_{p^*}(x_{n_k-1},\ x_{m_k}) < \epsilon. \tag{3.15}$$

By (3.14) and (3.15), we have

$$\begin{aligned}\epsilon &\le d_{p^*}(x_{n_k},\ x_{m_k})\\ &\le d_{p^*}(x_{n_k},\ x_{n_k-1}) + d_{p^*}(x_{n_k-1},\ x_{m_k})\end{aligned}$$

$$< d_{p^*}(x_{n_k}, x_{n_k-1}) + \epsilon.$$

Taking $\lim_{k\to\infty}$ on both sides in above inequality and from (3.13), we obtain

$$\lim_{k\to\infty} d_{p^*}(x_{n_k}, x_{m_k}) = \epsilon. \tag{3.16}$$

Using triangle inequality, we have

$$d_{p^*}(x_{n_k}, x_{m_k}) \leq d_{p^*}(x_{n_k}, x_{n_k-1}) + d_{p^*}(x_{n_k-1}, x_{m_k})$$

$$\leq d_{p^*}(x_{n_k}, x_{n_k-1}) + d_{p^*}(x_{n_k-1}, x_{m_k-1}) + d_{p^*}(x_{m_k-1}, x_{m_k})$$

and

$$d_{p^*}(x_{n_k-1}, x_{m_k-1}) \leq d_{p^*}(x_{n_k-1}, x_{n_k}) + d_{p^*}(x_{n_k}, x_{m_k-1})$$

$$\leq d_{p^*}(x_{n_k-1}, x_{n_k}) + d_{p^*}(x_{n_k}, x_{m_k}) + d_{p^*}(x_{m_k}, x_{m_k-1}).$$

Taking the limit as $k \to \infty$ in the above two inequalities and using (3.13) and (3.16), we get

$$\lim_{k\to\infty} d_{p^*}(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{3.17}$$

Now applying contractive condition (3.1), for $x_{n_k} \neq x_{m_k}$, we have

$$\varphi(| p^*(x_{n_k}, x_{m_k}) |) = \varphi(| p^*(\mathcal{T}x_{n_k-1}, \mathcal{T}x_{m_k-1}) |)$$

$$\leq \max\{\phi(| p^*(x_{n_k-1}, x_{m_k-1}) |),$$

$$\phi(| \frac{p^*(x_{n_k-1}, \mathcal{T}x_{n_k-1})(1 + p^*(x_{m_k-1}, \mathcal{T}x_{m_k-1}))}{1 + p^*(x_{m_k-1}, \mathcal{T}x_{m_k-1})} |)\}$$

$$= \max\{\phi(| p^*(x_{n_k-1}, x_{m_k-1}) |),$$

$$\phi(| \frac{p^*(x_{n_k-1}, x_{n_k})(1 + p^*(x_{m_k-1}, x_{m_k}))}{1 + p^*(x_{m_k-1}, x_{m_k} - 1)} |)\} \tag{3.18}$$

Let us put

$$E = \{n \in \mathbb{N} | \varphi(| p^*(x_{n_k}, x_{m_k}) |) \leq \phi(| p^*(x_{n_k -1}, x_{m_k -1}) |)\}$$

$$F = \{n \in \mathbb{N} | \varphi(| p^*(x_{n_k}, x_{m_k}) |) \leq \phi(| \frac{p^*(x_{n_k -1}, x_{n_k})(1 + p^*(x_{m_k -1}, x_{m_k}))}{1 + p^*(x_{n_k -1}, x_{m_k -1})} |)\}.$$

By (3.18), we have Card $E = \infty$ or Card $F = \infty$. Let us suppose that Card $E = \infty$, then there exists infinitely many $k \in \mathbb{N}$ satisfying

$$\varphi(| p^*(x_{n_k}, x_{m_k}) |) \leq \phi(| p^*(x_{n_k -1}, x_{m_k -1}) |) \tag{3.19}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, by letting the limit as $k \to \infty$, we have

$$\lim_{k \to \infty} | p^*(x_{n_k}, x_{m_k}) | \leq \lim_{k \to \infty} | p^*(x_{n_k -1}, x_{m_k -1}) |.$$

In the view of (3.16) and (3.17), we get $\epsilon = 0$ a contradiction. On the other hand, if Card $F = \infty$, then we can find infinitely many $k \in \mathbb{N}$ satisfying

$$\varphi(| p^*(x_{n_k}, x_{m_k}) |) \leq \phi\left(| \frac{p^*(x_{n_k -1}, x_{n_k})(1 + p^*(x_{m_k -1}, x_{m_k}))}{1 + p^*(x_{n_k -1}, x_{m_k -1})} | \right) \tag{3.20}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, we infer

$$| p^*(x_{n_k}, x_{m_k}) | \leq | \frac{p^*(x_{n_k -1}, x_{n_k})(1 + p^*(x_{m_k -1}, x_{m_k}))}{1 + p^*(x_{n_k -1}, x_{m_k -1})} |.$$

Taking the limit as $k \to \infty$ and in the view of (3.13) and (3.16), it follows that $\epsilon \leq 0$ and we reach a contradiction. Therefore, in both the possibilities, we reach a contradiction and therefore $\lim_{m,n \to \infty} d_{p^*}(x_n, x_m) = 0$. Similarly we can prove that $\lim_{m,n \to \infty} d_{p^*}(x_m, x_n) = 0$. Hence $\lim_{m,n \to \infty} d^s_{p^*}(x_n, x_m) = 0$, which ensures that $\{x_n\}$ is a Cauchy sequence in $(X, d^s_{p^*})$. Since (X, p^*) is a complete dualistic partial metric space, by Lemma 2.9(2), $(X, d^s_{p^*})$ is a complete metric space. Thus, there exists $v \in (X, d^s_{p^*})$ such that $x_n \to v$ as

$n \to \infty$, that is $\lim_{n\to\infty} d_{p^*}(x_n, v) = 0$ and by Lemma 2.9 (3), we know that

$$p^*(v, v) = \lim_{n\to\infty} p^*(x_n, v) = \lim_{n\to\infty} p^*(x_n, x_m). \tag{3.21}$$

Since, $\lim_{n\to\infty} d_{p^*}(x_n, v) = 0$, by (2.2), (3.7) and (3.12), we have

$$p^*(v, v) = \lim_{n\to\infty} p^*(x_n, v) = \lim_{n\to\infty} p^*(x_n, x_m) = 0. \tag{3.22}$$

This shows that $\{x_n\}$ is a Cauchy sequence converging to $v \in (X, p^*)$. We are left to prove that v is a fixed point of $\mathcal{T}$. Suppose that $\mathcal{T}v \neq v$. Now applying contractive condition (3.1) and Lemma 2.9(3), we have

$$\varphi(|\, p^*(\mathcal{T}v, \mathcal{T}x_n)\,|) \leq \max\left\{\phi(|\, p^*(v, x_n)\,|), \phi(|\, \frac{p^*(x_n, \mathcal{T}x_n)(1 + p^*(v, \mathcal{T}v))}{1 + p^*(x_n, v)}\,|)\right\}. \tag{3.23}$$

Denote

$$G = \{n \in \mathbb{N} \mid \varphi(|\, p^*(\mathcal{T}v, \mathcal{T}x_n)\,|) \leq \phi(|\, p^*(v, x_n)\,|)\}$$

$$H = \{n \in \mathbb{N} \mid \varphi(|\, p^*(\mathcal{T}v, \mathcal{T}x_n)\,|) \leq \phi(|\, \frac{p^*(x_n, \mathcal{T}x_n)(1 + p^*(v, \mathcal{T}v))}{1 + p^*(x_n, v)}\,|)\}.$$

We have Card $G = \infty$ or Card $H = \infty$. If Card $G = \infty$, then there exists infinitely many $n \in \mathbb{N}$ such that

$$\varphi(|\, p^*(\mathcal{T}v, \mathcal{T}x_n)\,|) \leq \phi(|\, p^*(v, x_n)\,|) \tag{3.24}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, by taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} |\, p^*(\mathcal{T}v, \mathcal{T}x_n)\,| \leq \lim_{n\to\infty} |\, p^*(v, x_n)\,|.$$

To simplify our consideration, we will denote the subsequence by the same symbol $\{\mathcal{T}x_n\}$ Since $\mathcal{T}x_n = x_{n+1}$ and $x_n \to v \in X$, this means that $\limsup p^*(v, x_n) \to 0$ and consequently $\lim_{n\to\infty} x_{n+1} = v$. We infer $|\, p^*(\mathcal{T}v, v)\,| \leq 0$ and then $|\, p^*(\mathcal{T}v, v)\,| = 0$. On the other hand, if Card $H = \infty$, then we can find infinitely many $n \in \mathbb{N}$, such that

$$\varphi(|\; p^*(\mathcal{T}v,\, \mathcal{T}x_n)\,| \le \phi(|\; \frac{p^*(x_n,\, \mathcal{T}x_n)(1 + p^*(v,\, \mathcal{T}v))}{1 + p^*(x_n,\, v)}\; |). \tag{3.25}$$

Since $(\varphi, \phi) \in \mathfrak{F}$, $\mathcal{T}x_n = x_{n+1}$ and $\lim_{n\to\infty} x_n = v$, on letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} |\; p^*(\mathcal{T}v,\, x_{n+1})\,| \le \lim_{n\to\infty} |\; \frac{p^*(x_n,\, x_{n+1})(1 + p^*(v,\, \mathcal{T}v))}{1 + p^*(x_n,\, v)}\; |. \tag{3.26}$$

In the view of (3.7), arguing like above, we conclude that $|\, p^*(\mathcal{T}v,\, v)\,| = 0$. Therefore, in both the cases, we obtain $|\, p^*(\mathcal{T}v,\, v)\,| = 0$ and then $p^*(\mathcal{T}v,\, v) = 0$. Since $\mathcal{T}$ has (CCP), we get

$$0 = p^*(v,\, v) \le kp^*(\mathcal{T}v,\, \mathcal{T}v) \tag{3.27}$$

On the other hand, by axiom (p_4^*) we have

$$p^*(v,\, v) \le p^*(v,\, \mathcal{T}v) + p^*(\mathcal{T}v,\, v) - p^*(\mathcal{T}v,\, \mathcal{T}v)$$

which implies that

$$p^*(\mathcal{T}v,\, \mathcal{T}v) \le 0. \tag{3.28}$$

The inequalities (3.27) and (3.28) imply that $p^*(\mathcal{T}v,\, \mathcal{T}v) = 0$. Thus

$$p^*(\mathcal{T}v,\, \mathcal{T}v) = p^*(v,\, v) = p^*(v,\, \mathcal{T}v). \tag{3.29}$$

By using axiom (p_1^*), we have $\mathcal{T}v = v$ and hence v is a fixed point of $\mathcal{T}$. Finally, we will prove the uniqueness of the fixed point. Suppose that $v^* \in X$ is another fixed point of $\mathcal{T}$ such that $v \neq v^*$. Now using contractive condition (3.1), we get

$$\varphi(|\; p^*(v,\, v^*)\,|) = \varphi(|\; p^*(\mathcal{T}v,\, \mathcal{T}v^*)\,|)$$

$$\le \max\left\{\phi(|\; p^*(v,\, v^*)\,|),\, \phi(|\; \frac{p^*(v,\, \mathcal{T}v^*)(1 + p^*(v,\, \mathcal{T}v^*))}{1 + p^*(v,\, v^*)}\; |)\right\}$$

$$= \max\left\{\phi(|\ p^*(v, v^*)\ |),\ \phi(|\ \frac{p^*(v, v^*)(1 + p^*(v, v))}{1 + p^*(v, v^*)}\ |)\right\}$$

$$= \max\{\phi(|\ p^*(v, v^*)\ |),\ \phi(0)\}. \tag{3.30}$$

If $\max\{\phi(|\ p^*(v, v^*)\ |), \phi(0)\} = \phi(|\ p^*(v, v^*)\ |)$, in this case from (3.30), $\varphi(|\ p^*(v, v^*)\ |) \leq \phi(|\ p^*(v, v^*)\ |)$. Since $(\varphi, \varphi) \in \mathfrak{F}$ and by Remark 2.18, we deduce that $|\ p^*(v, v^*)\ | = 0$. Similarly, if $\max\{\phi(|\ p^*(v, v^*)\ |), \phi(0)\} = \phi(0)$, then from (3.30), $\varphi(|\ p^*(v, v^*)\ |) \leq \phi(0)$. We infer that $|\ p^*(v, v^*)\ | \leq 0$ and then $|\ p^*(v, v^*)\ | = 0$. Hence in the both possibilities, $|\ p^*(v, v^*)\ | = 0$ and then $p^*(v, v^*) = 0$. Thus $p^*(v, v^*) = p^*(v, v) = p^*(v^*, v^*)$, by using axiom (p_1^*), we have $v = v^*$ and hence v is a unique fixed point of $\mathcal{T}$. This completes the proof.

From Theorem 3.1 we obtain the following corollaries.

Corollary 3.2. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying*

$$\varphi(|\ p^*(\mathcal{T}x, \mathcal{T}y)\ |) \leq \phi(|\ p^*(x, y)\ |) \tag{3.31}$$

$\forall x, y \in X$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Corollary 3.3. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying*

$$\varphi(|\ p^*(\mathcal{T}x, \mathcal{T}y)\ | \leq \phi(|\ \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)}\ |) \tag{3.32}$$

$\forall x, y \in \Delta$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Taking into account Example 2.21, we have the following corollary.

Corollary 3.4. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ satisfying*

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) | \le \max \{\varphi(| p^*(x, y) | - \phi(| p^*(x, y) |)),$$

$$\varphi(| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} | - \phi| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} |)\} \tag{3.33}$$

for all $x, y \in X$, where φ is an increasing function and ϕ is a non-decreasing function and they satisfy $\varphi(t) = \phi(t) = 0$ if and only if $t = 0$ and φ is continuous with $\phi \le \varphi$. $\forall x, y \in X$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Corollary 3.4 has the following consequences.

Corollary 3.5. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ satisfying the same conditions as in Corollary 3.4*

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) \le \phi(| p^*(x, y) | - \phi| p^*(x, y) |) \tag{3.34}$$

for all $x, y \in X$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Corollary 3.6. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ satisfying the same conditions as in Corollary 3.4.*

$$\varphi(|\ p^*(\mathcal{T}x,\ \mathcal{T}y)\ |) \leq \varphi(|\ \frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\ |$$

$$- \phi|\ \frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\ |) \tag{3.35}$$

for all $x,\ y \in X$. *If* $\mathcal{T}$ *satisfies* (*CCP*). *Then* $\mathcal{T}$ *has a unique fixed point in* X *and the Picard iterative sequence* $\{\mathcal{T}^n(x_0)\}$ *with initial point* x_0, *converges to the fixed point.*

Remark 3.8. The main result of [26] is Theorem 2.10. Notice that the rational contractive condition appearing in this theorem

$$|\ p^*(\mathcal{T}x,\ \mathcal{T}y)\ | \leq \alpha|\ \frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\ | + \beta|\ p^*(x,\ y)\ |$$

for any $x,\ y \in \Delta$, where $\alpha,\ \beta \geq 0$ and $\alpha + \beta < 1$ implies that

$$|\ p^*(\mathcal{T}x,\ \mathcal{T}y)\ | \leq (\alpha + \beta)\max\left\{\phi|\ \frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\ |,\ |\ p^*(x,\ y)\ |\right\}$$

$$\leq \max\left\{(\alpha+\beta)|\ \frac{p^*(y,\mathcal{T}y)(1+p^*(x,\mathcal{T}x))}{1+p^*(x,y)}\ |,(\alpha+\beta)|\ p^*(x,y)\ |\right\}.$$

This condition is a particular case of the contractive condition appearing in Theorem 3.1 with the pair of functions $(\varphi,\ \phi) \in \mathfrak{F}$ given by $\varphi = 1_{[0,\ \infty)}$ and $\phi = (\alpha + \beta)1_{[0,\ \infty)}$. Therefore, Theorem 2.10 is a particular case of the following corollary and considered as an extension and generalizations of Theorem 2.10 in the setting of complete dualistic partial metric spaces.

Corollary 3.9. *Let* $(X,\ p^*)$ *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a mapping such that*

$$|\ p^*(\mathcal{T}x,\mathcal{T}y)\ | \leq \max\left\{(\alpha+\beta)|\ \frac{p^*(y,\mathcal{T}y)(1+p^*(x,\mathcal{T}x))}{1+p^*(x,y)}\ |,(\alpha+\beta)|\ p^*(x,y)\ |\right\} \tag{3.36}$$

for any $x,\ y \in X$, *where* $\alpha,\ \beta \geq 0$ *and* $\alpha + \beta < 1$. *If* $\mathcal{T}$ *satisfies* (*CCP*). *Then*

$\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Observations 3.10.

1. If in Corollary 3.9, we put $\alpha + \beta = c$ and $\max\left\{\left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \right|, | p^*(x, y) |\right\} = | p^*(x, y) |,$ then we get Theorem 2.3 of Oltra and Valero [28].

2. In Corollary 3.9, if we replace the range of p^* by $[0, \infty)$, put $\alpha + \beta = c$ and $\max\left\{\left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \right|, | p^*(x, y) |\right\} = | p^*(x, y) |,$ then we get fixed point theorem of Matthews [17].

3. If we set $p^*(x, x) = 0, \forall x \in X$ and replace the range of p^* by $[0, \infty)$, in Theorems 3.1, we retrieve corresponding theorems in metric spaces (see [8]).

4. If we set $p^*(x, x) \in [0, \infty), \forall x, y \in X$ in Theorems 3.1, we retrieve corresponding theorems in partial metric spaces (see [33]).

Taking into account Example 2.20, we have the following corollary.

Corollary 3.11. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exist $\ell \in \mathcal{S}$ (see Example 2.20) satisfying*

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) \leq \max\{\ell(| p^*(x, y) |)| p^*(x, y) |,$$
$$\ell\left(\left| \frac{p^*(y, \mathcal{T}y)(1+p^*(x, \mathcal{T}x))}{1+p^*(x, y)} \right|\right)\left| \frac{p^*(y, \mathcal{T}y)(1+p^*(x, \mathcal{T}x))}{1+p^*(x, y)} \right|\} \quad (3.37)$$

for all $x, y \in \Delta$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Following Corollary is a generalization of main result of Geraghty [9].

Corollary 3.11. *Let* (X, p^*) *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a mapping such that there exist* $\ell \in \mathcal{S}$ *(see Example* 2.20) *satisfying*

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) \leq \ell(| p^*(x, y) |) | p^*(x, y) | \tag{3.38}$$

for all $x, y \in X$. *If* $\mathcal{T}$ *satisfies* (*CCP*). *Then* $\mathcal{T}$ *has a unique fixed point in* X *and the Picard iterative sequence* $\{\mathcal{T}^n(x_0)\}$ *with initial point* x_0, *converges to the fixed point.*

4. Examples

In this section, we give an example in support of our main result.

Example 4.1. Let $X = (-\infty, 0]^2$. Define $p^* : X \times X \to (-\infty, \infty)$ by $p^*(x, y) = \max\{x_1, y_1\}$ where $x = (x_1, y_1)$ and $y = (x_2, y_2)$. It is easy to check that $((-\infty, 0]^2, p^*)$ is a complete dualistic partial metric space. Define $\mathcal{T} : (-\infty, 0]^2 \to (-\infty, 0]^2$ by $\mathcal{T}x = x^2, \forall x \in (-\infty, 0]^2$. Since

$$\max\{x_1, y_1\} \leq \max\left\{\frac{x_1}{2}, \frac{y_1}{2}\right\} \Rightarrow p^*(x, y) \leq p^*(\mathcal{T}x, \mathcal{T}y), \forall x, y \in (-\infty, 0]^2.$$

Hence $\mathcal{T}$ satisfies (CCP). Define $\varphi, \phi : [0, \infty) \to [0, \infty)$ as follows: $\varphi(t) = \ln\left(\frac{5t+1}{2}\right)$ and $\varphi(t) = \ln\left(\frac{3t+1}{2}\right), \forall t \in [0, \infty)$. Clearly, $(\varphi, \phi) \in \mathfrak{F}$. We shall show (3.1) is satisfied. Without loss of generality, assume that $x_1 \leq y_1$. Then we have

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) = \ln\left(\frac{5| p^*(\mathcal{T}x, \mathcal{T}y) | + 1}{12}\right)$$

$$= \ln\left(\frac{5|\ p^*\left(\frac{x}{2}, \frac{y}{2}\right)| + 1}{12}\right) = \ln\left(\frac{5|\left(\frac{y_1}{2}\right)| + 1}{12}\right)$$

$$= \ln\left(\frac{5}{24}|\ y_1\ | + \frac{1}{12}\right).$$

On the other hand,

$$\phi(|\ p^*(x,\ y)\ |) = \ln\left(\frac{3|\ p^*(x,\ y)\ | + 1}{12}\right) = \ln\left(\frac{3|\ y_1\ | + 1}{12}\right) = \ln\left(\frac{3}{12}|\ y_1\ | + \frac{1}{12}\right)$$

$$\phi\left(|\ \frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\right) = \phi\left(|\ \frac{\frac{y_1}{2}\left(1 + \frac{x_1}{2}\right)}{1 + y_1}\right) = \left(|\ \frac{y_1(2 + x_1)}{4(1 + y_1)}\ |\right)$$

$$= \ln\left(|\ \frac{3|\ \frac{y_1(2 + x_1)}{4(1 + y_1)}\ | + 1}{12}\right)$$

$$= \ln\left(\frac{3|\ y_1(2 + x_1)\ | + 4|\ 1 + y_1\ |}{24}\ |\right).$$

Combining the observations above, we get

$$\varphi(p^*(\mathcal{T}x,\ \mathcal{T}y)\ |) = \ln\left(\frac{5}{24}|\ y_1\ | + \frac{1}{12}\right) = \ln\left(\frac{3}{12}|\ y_1\ | + \frac{1}{12}\right)$$

$$\leq \max\left\{\ln\left(\frac{3}{12}|\ y_1\ | + \frac{1}{12}\right),\ \ln\left(\frac{3|\ y_1(2 + x_1)\ | + 4|\ 1 + y_1\ |}{24}\right)\right\}$$

$$= \max\left\{\phi(p^*(x,\ y)\ |),\ \phi\left(\frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\ |\right)\right\}.$$

Thus all the conditions of Theorem 3.1 are satisfied. Hence $\mathcal{T}$ has a fixed point, indeed $v = (0,\ 0)$ is a fixed point.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors' Contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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ON THE NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

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ABSTRACT

In this work we concern with the approximate solution of the linear equation $Af = f$ where A is injective and compact operator, this equation admits a unique solution in direct sense or in the least square sense provided the right-hand side f is in R (A) or in $R(A) + R(A)\perp$ respectively. the solution is not stable. Besides, if A is positive de.nite we can replace the original equation by the auxiliary one $\alpha\varphi_\alpha + A\varphi_\alpha = f$ *where its solution* φ_α *exist, stable and converges to the exact solution* φ *of the original equation as* α *tends to zero.*

Keywords: first-kind Fredholm integral equations, Lavrentiev method, Tikhonov regularization, numerical quadrature, discrete approximation.

1. Introduction

The inverse problem takes a considerable part in the domain of differential and partial differential equations. As prominent example of the ill posed problem we find integral equation of the first kind where the most problems of engineering and mathematical physics can be modelled in this equation, many methods study its approximation solution and stability. In [1] the authors solved the integral equation of the first kind by Chebyshev wavelet constructed in the bounded interval and used the Galerkin technical in order to reduce the integral equation into an algebraic linear system. On the other hand, in [3, 4] the collocation method with Legendre wavelets are used directly for this equation and convert it to an algebraic linear system.

The use of the since basis functions method for solving the first kind integral equation was find in [5]. The method proposed by authors in [10] is to use the Hermite polynomial with least square method in order to solve integral equation of the first kind with degenerate kernel supported by Galerkin and collocation methods. Let A be a linear compact operator de.ned from Hilbert space H to itself over the field $\mathbb{R}$ We explicit the linear inverse problem of a first kind by

$$A\varphi = f, \quad (1)$$

where f is the data function and φ the unknown potential one, suppose that A is injective, then the equation (1) admits a unique solution in direct sense or in the last square sense provided the right-hand side f is in $R(A)$ or in $R(A) + R(A)^{\perp}$, respectively. Due to the nonclosed range $R(A)$ the solution is not stable. Besides, if A is positive definite we can replace the original equation by the auxiliary one

$$\alpha\varphi_\alpha + A\varphi_\alpha = f, \tag{2}$$

where we add the term $\alpha\varphi$ to the operator $A\varphi$ for α positive and small, the equation (2) admits a stable solution φ_α. Noting that the function φ_α converges to the exact solution φ of equation (1) as α tends to zero [8].

Lavrentiev method

The Lavrentiev method for the equation (1) is to replace the equation by the following one $A_\varphi = f_\delta$, with $\| f - f_\delta \| \leq \delta$, if the right-hand side f_δ is not in the range $R(A)$, Lavrentiev changes the equation $A\varphi = f_\delta$ by its auxiliary equation

$$\alpha\varphi_{\alpha\delta} + A\varphi_{\alpha\delta} = f_\delta, \ \alpha > 0. \tag{3}$$

It is clear that, if the operator A is positive definite, the problem of the second kind (3) is well-posed. Lavrentiev proves that the solution $\varphi_{\alpha\delta}$ of the equation (3) tends to the exact solution φ of the equation (1) with conditions α and δ tend to zero.

Tikhonov Regularization Method

The Tikhonov regularization of the equation (1) corresponds to the regularization operators

$$R_\alpha = (\alpha I + A^* A)^{-1} A^*, \text{ for } \alpha > 0, \tag{4}$$

which approximates the unbounded operator A^{-1} on $R(A)$. Noting that, the solution $\varphi_\alpha = R_\alpha f$ represents the unique solution of the equation

$$\alpha\varphi_\alpha + A^* A\varphi_\alpha = A^* f, \tag{5}$$

and depends continuously on f, for all $f \in H$ and $\alpha > 0$. Also this solution is the unique minimum of the Tikhonov functional

$$J_\alpha(X) = \| A\varphi - f \|^2 + \alpha \| \varphi \|^2, \text{ for } \varphi \in H \text{ and } \alpha > 0.$$

2. Main Results

In this work we focus our study to the Fredholm integral equations of the first kind

$$A\varphi(x) = \int_a^b k(x, t)\varphi(t)dt = f(x),\ a \le x \le b$$

where $k(x, t)$ and f are given continuous functions and $\varphi(x) \in H([a, b])$ is the unknown potential function to be determined.

Lemma 1 [9]. *The problem* (2) *is well posed with the norm* $\| (\alpha I + A)^{-1} \| = O\left(\frac{1}{\sqrt{\alpha}}\right)$ *provided* A *is injective and positive definite operator*.

Proposition. *The injectivity and the positivity of the compact operator A lead to the existence and uniqueness of the solution of the auxiliary problem* (2)

$$\alpha\varphi_\alpha + A\varphi_\alpha = f,\ \alpha > 0.$$

Besides, the solution φ_α *converges to the exact solution* φ *of the initial problem* (1), *as* α *goes to zero, say*

$$\lim_{\alpha \to 0} \| \varphi - \varphi_\alpha \| = 0.$$

Proof.

Indeed,

$$\varphi - \varphi_\alpha = \varphi - (\alpha I + A)^{-1} f$$

$$= \varphi - (\alpha I + A)^{-1} A\varphi$$

$$= \alpha(\alpha I + A)^{-1}\varphi.$$

Therefore

$$\| \varphi - \varphi_\alpha \| \le \alpha \| (\alpha I + A)^{-1} \| \| \varphi \|$$

$$\| \varphi - \varphi_\alpha \| = O(\sqrt{\alpha}).$$

Nyström method

Using the quadrature rule to approximate $\int_a^b k(x, t)\varphi(t)dt$ say

$$A\varphi(x) = \int_a^b k(x, t)\varphi(t)dt \simeq \sum_{j=1}^{n} w_j k(x, t_j)\varphi(t_j).$$

So, the equation (1) can be replaced by

$$A_n\varphi(x) = \sum_{j=1}^{n} w_j k(x, t_j)\varphi(t_j) = f(x), \ 0 \le x \le 1. \tag{6}$$

In the collocation method the values of $\varphi(t_j)$, $j = 1, 2, \ldots, n$ are found so that the equation (6) is verified for all points $x_1, x_2, \ldots, x_m$, in [0, 1]. It is not necessary to take $m = n$, but often m and n are chosen to be equal, and x_i is chosen as $x_i = t_i$, $i = 1, 2, \ldots, n$.

$$\sum_{j=1}^{n} w_j k(x, t_j)\varphi(t_j) = f(x_i), \ i = 1, 2, \ldots, n. \tag{7}$$

Taking $\mathcal{A} = (a_{ij})$ the $n \times n$ matrix such that $a_{ij} = w_j k(x_i, t_j)$ for $1 \le i, j \le n$, the unknown vector $\vec{\Phi} = (\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_n))^T = (\varphi_1, \varphi_2, \ldots, \varphi_n)^T$ and the right-hand side vector $\vec{F} = (f(x_1), f(t_2), \ldots, f(x_n))^T = (f_1, f_2, \ldots, f_n)^T$, Then the auxiliary equation (2) can be approximated by the matrix equation

$$(\alpha I + \mathcal{A})\overrightarrow{\Phi_\alpha} = \vec{F}. \tag{8}$$

The algebraic system (8) admits a unique solution $\overrightarrow{\Phi_\alpha}$ converges to the solution $\vec{\Phi}$ of the system $\mathcal{A}\overrightarrow{\Phi_\alpha} = \vec{F}$ as $\alpha \to 0$.

Lemma 2. *The norm* $\| (\alpha I + A)^{-1} \| \le \frac{1}{\alpha}$ *provided A is positive definite operator. Indeed, A is positive and injective, it follows*

$$\langle \varphi, \varphi \rangle = \langle (\alpha I + A)(\alpha I + A)^{-1}\varphi, (\alpha I + A)(\alpha I + A)^{-1}\varphi \rangle$$

$$= \langle \alpha(\alpha I + A)^{-1}\varphi + A(\alpha I + A)^{-1}\varphi, \alpha(\alpha I + A)^{-1}\varphi + A(\alpha I + A)^{-1}\varphi \rangle$$

$$= \| \varphi \|^2 = \alpha^2 \| (\alpha I + A)^{-1}\varphi \|^2 + 2\alpha\langle A(\alpha I + A)^{-1}\varphi, (\alpha I + A)^{-1}\varphi \rangle$$

$$+ \| A(\alpha I + A)^{-1}\varphi \|^2, \| \varphi \|^2 \geq \alpha^2 \| (\alpha I + A)^{-1}\varphi \|^2.$$

Therefore, we obtain

$$\| (\alpha I + A)^{-1} \| \leq \frac{1}{\alpha}.$$

Lemma 3. *The operator* $(\alpha I + A_n)$ *is invertible on the Hilbert space H to itself if* $\| (A - A_n)A_n \| \leq \alpha^2$. *Besides, if* $(\alpha I + A_n)\varphi_n = f$ *and* $(\alpha I + A)\varphi = f$, *then*

$$\| \varphi - \varphi_n \| = O(\alpha).$$

Indeed,

$$\frac{1}{\alpha}[I - (\alpha I + A)^{-1}A_n](\alpha I + A_n) = I - (\alpha I + A)^{-1}A_n + \frac{1}{\alpha}[I - (\alpha I + A)^{-1}A_n]A_n$$

$$= I - (\alpha I + A)^{-1}A_n + \frac{1}{\alpha}[(\alpha I + A)^{-1}((\alpha I + A) - A_n)]A_n$$

$$= I - (\alpha I + A)^{-1}A_n + \frac{1}{\alpha}(\alpha I + A)^{-1}(\alpha I + A - A_n)A_n$$

$$= \left(I + \frac{1}{\alpha}(\alpha I + A)^{-1}(A - A_n)A_n \right).$$

Noting that, the right side of the last expression is invertible, for

$$\frac{1}{\alpha}\| (\alpha I + A)^{-1}(A - A_n)A_n \| \leq \frac{1}{\alpha}\| (\alpha I + A)^{-1} \| \| (A - A_n)A_n \| < 1.$$

The injectivity of the right side involves the one of $(\alpha I + A_n)$ and so its bijectivity. For the error, we get

$$\| \varphi_n - \varphi \| = \| ((\alpha I + A_n)^{-1} - (\alpha I + A)^{-1} f \|$$

$$= \| ((\alpha I + A)^{-1}(A_n - A)(\alpha I + A_n)^{-1})f \|$$

$$= \| ((\alpha I + A)^{-1}(A - A_n)\varphi_n \|$$

$$\leq \| (\alpha I + A)^{-1} \| \| (A - A_n) \| \| \varphi_n \|$$

$$= O(\alpha).$$

3. Explanatory Examples

Example 1. Consider the first-kind integral equations of Fredholm

$$\int_0^1 \exp(xt)\varphi(t)dt = \frac{\exp(x+1) - 1}{x+1},$$

where $0 \leq t, x \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by

$$\varphi(x) = \exp(x).$$

Table 1. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 1 in some arbitrary points for $N = 10$ and $\alpha = 10^{-6}$, the error is compared with the Chebyshev Wavelet Method [1] and the Haar wavelets method [6].

Val of x	Exact sol φ	App sol φ_α	$\vert \varphi - \varphi_\alpha \vert$	Error [1]	Error [6]
0.000	1.00e+00	9.99e-01	1.21e-06	1.46e - 05	7.85e-03
0.200	1.22e+00	1.22e+00	8.22e-07	1.73e - 05	5.69e-03
0.400	1.49e+00	1.49e+00	3.09e-07	1.57e - 05	2.31e-03
0.600	1.82e+00	1.82e+00	1.05e-06	1.30e - 06	2.86e-03

0.800	2.22e+00	2.22e+00	1.80e-06	1.52e - 05	1.04e-02
1.000	2.71e+00	2.71e+00	7.95e-06	1.04e - 05	4.98e-03

Example 2. Consider the first-kind integral equations of Fredholm

$$\int_0^1 \exp(t \sin x)\varphi(t)dt = \frac{1}{1+\sin^2 x}(\exp(\sin x)(\cos 1 \sin x + \sin 1) - \sin x),$$

where $0 \leq t, x \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) = \cos x$.

Table 2. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 2 in some arbitrary points for $N = 10$ and $\alpha = 10^{-6}$, the error is compared with Legendre wavelets collocation method [3].

Val of x	Exact sol φ	App sol φ_α	$\lvert\varphi - \varphi_\alpha\rvert$	Error [3]
0.000	1.000e+00	9.991e-01	8.828e-04	1.488e-03
0.200	9.800e-01	9.803e-01	2.667e-04	2.531e-03
0.400	9.210e-01	9.207e-01	3.320e-04	4.740e-03
0.600	8.253e-01	8.255e-01	2.007e-04	2.920e-03
0.800	6.967e-01	6.966e-01	4.073e-05	4.951e-03
0.900	5.816e-01	5.815e-01	9.163e-05	1.239e-03

Example 3. Consider the first-kind integral equations of Fredholm

$$\int_0^1 (x^2 + 2tx + t^2)\varphi(t)dt = \frac{x^2}{2} + \frac{2x}{3} + \frac{1}{4},$$

where $0 \leq t, x \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) = x$.

Table 3. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 3 in some arbitrary points for $N = 10$ and $\alpha = 10^{-6}$, the error is compared with Hermite polynomial method [9].

Val of x	Exact sol φ	App sol φ_α	$\|\varphi - \varphi_\alpha\|$	Error [10]
0.000	0.000e+00	1.781e-07	1.781e-07	1.6e-03
0.200	2.000e-01	2.000e-01	2.158e-08	6.0e-04
0.400	4.000e-01	4.000e-01	4.687e-08	5.0e-04
0.600	6.000e-01	5.999e-01	8.356e-08	5.0e-04
0.800	8.000e-01	8.000e-01	7.016e-09	4.0e-04
1.000	1.000e+00	1.000e+00	1.140e-07	1.7e-03

Example 4. Consider the first-kind integral equations of Fredholm

$$\int_0^1 \cos(x - t)\varphi(t)dt = \frac{1}{4}\cos x + \frac{1}{2}\sin x - \frac{1}{4}\cos(x - 2),$$

where $0 \leq t, x \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) = \sin x$.

Table 4. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 4 in some arbitrary points, the error is calculated for $N = 10$ and $\alpha = 10^{-6}$,

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Table 4. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 4 in some arbitrary points, the error is calculated for $N = 10$ and $\alpha = 10^{-6}$.

Val of x	Exact sol φ	App sol φ_α	$\lvert \varphi - \varphi_\alpha \rvert$
0.000	0.000e+00	1.384e-08	1.384e-08
0.200	1.986e-01	1.986e-01	8.067e-09
0.400	3.894e-01	3.894e-01	5.170e-09
0.600	5.646e-01	5.646e-01	1.680e-08
0.800	7.173e-01	7.173e-01	1.941e-08
1.000	8.414e-01	8.414e-01	3.261e-08

Example 5. Consider the first-kind integral equations of Fredholm

$$\int_0^1 \sinh(x - t)\varphi(t)dt = -\frac{1}{8}\exp(x)(\exp(-2) - 3 + \exp(-2x) + \exp(2)\exp(-2x)),$$

where $0 \leq t, x \leq 1$, and the function $f(x)$ is chosen so that the exact solution is given by $\varphi(x) \cosh x$.

Table 5. We present the exact solution φ and its approximate one φ_α as well as the absolute error $|\varphi - \varphi_\alpha|$ of the example 5 in some arbitrary points, the error is calculated for $N = 10$ and $\alpha = 10^{-6}$,

Val of x	Exact sol φ	App sol φ_α	$\lvert \varphi - \varphi_\alpha \rvert$
0.000	1.000e+00	9.999e-01	5.193e-08
0.200	1.020e+00	1.020e+00	3.413e-08

0.400	1.081e+00	1.081e+00	2.062e-08
0.600	1.185e+00	1.185e+00	1.546e-09
0.800	1.337e+00	1.337e+00	2.289e-08
1.000	1.543e+00	1.543e+00	2.962e-08

4. Conclusion

This numerical technique for solving Fredholm integral equations of first kind, concentrated on the few modification of Lavrentiev classical method supported by the modified Simpson approximation [7], the approximate solution $\varphi\alpha$ is measurably close to the exact solution φ of the given equation on the whole interval [0, 1]. This method is tested by solving some examples for which the exact solution is known and proves its efficiency compared with other methods.

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