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Contents

Inclusion Results of a Generalized Mittag-Leffler-Type Poisson Distribution in the k-Uniformly Janowski Starlike and the k-Janowski Convex

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A B S T R A C T

Due to the Mittag-Leffler function's crucial contribution to solving the fractional integral and differential equations, academics have begun to pay more attention to this function. The Mittag-Leffler function naturally appears in the solutions of fractional-order differential and integral equations, particularly in the studies of fractional generalization of kinetic equations, random walks, Levy flights, super-diffusive transport, and complex systems. As an example, it is possible to find certain properties of the Mittag-Leffler functions and generalized Mittag-Leffler functions [4,5]. We consider an additional generalization in this study, $E_{\alpha,\beta}^{\theta}(z)$, *given by Prabhakar [6,7]. We normalize the later to deduce* $\mathbb{E}_{\alpha,\beta}^{\theta}(z)$ *in order to explore the inclusion results in a well-known class of analytic functions, namely* $k-\mathcal{ST}[A,B]$ *and* $k-\mathcal{UCV}[A,B]$, *k-uniformly Janowski starlike and k-Janowski convex functions, respectively. Recently, researches on the theory of univalent functions emphasize the crucial role of implementing distributions of random variables such as the negative binomial distribution, thegeometric distribution, the hypergeometric distribution, and in this study, the focus is on the Poisson distribution associated with the convolution (Hadamard product) that isapplied to define and explore the inclusion results of the followings:* $\mathrm{I}_{\alpha,\beta}^{m,\theta}(z)$, $\mathcal{J}_{\alpha,\beta}^{m}f$ *and the integral operator* $\mathcal{G}_{\alpha,\beta}^{m,\theta}$. *Furthermore, some results of special cases will be also investigated.*

Keywords : k - Uniformly Janowski Star-like, k-Janowski Convex Functions, Mittag-Leffler Function

1. Introduction

In recent years, there has been a lot of interest in random variable distributions. In statistics and probability theory, the real variable x and the complex variable z's probability density functions been crucial. The distributions have so been thoroughly investigated. Many different types of distributions, including the negative geometric distribution, hypergeometric distribution, Poisson distribution, and binomial distribution, have been developed as a result of real-world events.

If a random variable's function of probability density is given by, then the variable x has a Poisson distribution:

$$f(x) = \frac{e^{-m}}{x!} m^x, x = 0,1,2, \dots \qquad (1.1)$$

For the parameter of the distribution m, the Poisson distribution started receiving interest in the theory of univalent functions, firstly by Porwal [8] and then later by Porwal and Dixit [9] who provided moments and moments' generating functions with the Mittag-Leffler Poisson distribution.

We indicate by $\mathcal{A}$ the well-known type of the form normalized functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.2)$$

Functions that in the open unit disk analyzers $\mathbb{U} = \text{i}\{z \in \mathbb{C}: |z| < 1\}$.

We also let $\mathcal{T}$ a sub-class of $\mathcal{A}$ that includes operations of the form

$$(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \ z \in \mathbb{U}. \qquad (1.3)$$

Now, we recall the definitions of the classes $k - \mathcal{ST}[A, B]$ and $k - \mathcal{UCV}[A, B]$ that were introduced and studied by Noor and Malik [4].

A function $f \in \mathcal{A}$ is considered to be a member of the class of k -Janowski star-like functions.

$k - \mathcal{ST}[A, B], k \geq 0, -1 \leq B < A \leq 1,$ if and only if

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)} - 1\right|. \quad (1.4)$$

Further, a function $f \in \mathcal{A}$ is said to be in the class k - Janowski convex functions $\mathcal{UCV}[A, B], k \geq 0, -1 \leq B\text{i} < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)} - 1\right|, (1.5)$$

clearly

$$f(z) \in k - \mathcal{UCV}[A, B] \Leftrightarrow zf'(z) \in k - \mathcal{ST}[A, B].$$

The above are generalizations of the following special cases:

(2) $k - \mathcal{ST}[1 - 2\gamma, -1] = k - \mathcal{SD}[k, \gamma]$ and $k - \mathcal{UCV}[1 - 2\gamma, -1] = k - \mathcal{KD}[k, \gamma]$, the classes introduced by Shams et al. in [10].

(3) $0 - \mathcal{ST}[A, B] = S^*[A, B]$ and $0 - \mathcal{UCV}[A, B] = \mathcal{C}[A, B]$ the well-known classes of Janows - ki starlike and Janowski convex functions respectively, introduced by Janowski [12].

(4) $0 - \mathcal{ST}[1 - 2\gamma, -1] = \mathcal{S}^*(\gamma)$ and $0 - \mathcal{UCV}[1 - 2\gamma, -1] = \mathcal{C}(\gamma)$, the well-known classes of starlike functions of order $\gamma(0 \leq \gamma < 1)$ and convex functions of order $\gamma(0 \leq \gamma < 1)$ respectively, (see [3]).

If $f(z) \in k - \mathcal{ST}[A, B]$ then

$$w = \frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}$$

takes all values from the domain $\Omega_k, k \geq 0$

$$\begin{aligned}\Omega_k &= \{w : \Re w > k|w-1|\} \\ &= \left\{u+iv : u > k\sqrt{(u-1)^2+v^2}\right\}\end{aligned}$$

The domain Ω_k represents the right half plane for $k=0;$ a hyperbola for $0< k< 1$; a parabola for $k=1$ and an ellipse for $k > 1$, (see [4]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(C,D), \tau \in \mathbb{C} \setminus \mathrm{i}\{0\}, -1 \leq D < C \leq 1,$ if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(C-D)\tau - D[f'(z)-1]}\right| < 1,\ z \in \mathbb{U}$$

The class above was introduced by Dixit and Pal [13] providing the below results

Lemma 1.1. [13] If $f \in \mathcal{R}^\tau(C,D)$ is of the form (1.2), then

$$|a_n| \leq (C-D)\frac{|\tau|}{n},\ n \in \mathbb{N} \setminus \{1\}$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(C-D)|\tau|t^{n-1}}{1+Dt^{n-1}}\right)dt,\ (z \in \mathbb{U} \in \mathbb{N} \setminus \{1\}).$$

Mittag-Leffler function $E_\alpha(z)$ is studied by Mittag-Leffler [2] and given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)},\ (z \in \mathbb{C}, \Re(\alpha) > 0).$$

Prabhakar [5, 11] has generalized the Mittag – Leffler function as follows

$$E_{\alpha,\beta}^{\theta}(z) := \sum_{n=0}^{\infty} \frac{(\theta)_n}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!},\quad z,\beta,\theta \in \mathbb{C}\,;\ \Re(\alpha) > 0\,,$$

here; $(\theta)_v$ denotes the familiar Pochhammer symbol defined as

$$(\theta)_v := \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases}1, & if\ v=0, \quad \theta \in \mathbb{C}\backslash\{0\} \\ \theta(\theta+1)\ldots(\theta+n-1), & if\ v=n \in N, \theta \in \mathbb{C}\end{cases}$$

$$(1)_n = n!, \qquad n \in N_0, N_0 = N \cup \{0\}, \qquad N = \{1,2,3,\ldots\}.$$

Since the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\theta}(z)$ doesn't belong to the family $\mathcal{A}$. Let us consider the following normalization of the Mittag-Leffler function

$$\mathbb{E}_{\alpha,\beta}^{\theta}(z) = \Gamma(\beta) z E_{\alpha,\beta}^{\theta}(z)$$
$$= z + \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta)}{n!\Gamma(\alpha(n-1)+\beta)} z^n \quad (1.6)$$

where $z, \alpha, \beta \in \mathbb{C}; \beta \neq 0, -1, -2, \cdots$ and $\Re(\beta) > 0, \Re(\alpha) > 0$.

Our attention in this paper is only to the cases; where α, β are real-valued and $z \in \mathbb{U}$.

The generalized Mittag-Leffler-type Poisson distribution's probability mass function is then given by

$$P(x = r) = \frac{m^r}{\Gamma(\alpha k + \beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)}, \; r = 0,1,2,3,\cdots,$$

in where m>0, α>0, and β>0. One can introduce a power series whose coefficients are probabilities of the generalized Mittag-Leffler-type Poisson distribution series using the normalized version of the Mittag-Leffler function in (1.6), as follows:

$$H_{\alpha,\beta}^{m,\theta}(z) := z + \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)} z^n, \; z \in \mathbb{U}$$

To serve our purpose, we also need to define the series

$$I_{\alpha,\beta}^{m,\theta}(z) := 2z - H_{\alpha,\beta}^{m}(z) = z - \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)} z^n, \; z \in \mathbb{U} \quad (1.7)$$

Finally, and by the means of the convolution, we deduce the following operator:

$$\mathcal{J}_{\alpha,\beta}^{m,\theta} f(z) = H_{\alpha,\beta}^{m,\theta}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)} a_n z^n, \; z \in \mathbb{U},$$

2. Inclusion Results of $I_{\alpha,\beta}^{m,\theta}(z)$

To establish our primary findings, we shall require the below given lemmasi

Lemma 2.1. *[4] A function f of the form (1.2) is in the class $k - \mathcal{ST}[A, B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|]|a_n| \leq |B - A| \quad (2.1)$$

where $-1 \leq B < A \leq 1$ and $k \geq 0$.

Lemma 2.2. *[4] A function f of the form (1.2) is in the class $k - \mathcal{UCV}[A, B]$, if it satisfies the condition*

$$\sum_{n=2}^{\infty} n[2(k+1)(n-1) + |n(B+1) - (A+1)|]|a_n| \leq |B - A \quad (2.2)$$

where $-1 \leq B < A \leq 1$ and $k \geq 0$.

In this study, we will assume that until otherwise stated that $\alpha, m > 0, k \geq 0$ and $-1 \leq B < A \leq 1$.

Theorem 2.3. *Let $\beta > 1$. Then $I_{\alpha,\beta}^{m,\theta} \in k - \mathcal{ST}[A, B]$ if*

$$\begin{aligned} &\frac{(\theta)_n \Gamma(\beta)}{n!\mathbb{E}_{\alpha,\beta}^{\theta}(m)} \Big[\frac{2k+B+3}{\alpha}\Big(E_{\alpha,\beta-1}(m) - \frac{1}{\Gamma(\beta-1)}\Big) \\ &+ \Big[\Big(\frac{2k+B+3}{\alpha}\Big)(1-\beta) + (B+A+2)\Big]\Big(E_{\alpha,\beta}^{\theta}(m) - \frac{n!}{(\theta)_n\Gamma(\beta)}\Big)\Big] \\ \leq \;& |B - A| \end{aligned} \quad (2.3)$$

Proof. Given Lemma 2.1 and (2.1), it is sufficient to demonstrate that

$$J_1 := \sum_{n=2}^{\infty} \left[2(k+1)(n-1) + |n(B+1) - (A+1)|\right] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{\mathrm{n}!\, \Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}(m)} \leq |B-A|$$

We have

$$J_1 \leq \sum_{n=2}^{\infty} \left[2(k+1)(n-1) + n(B+1) + (A+1)\right] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{\mathrm{n}!\, \Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$= \sum_{n=2}^{\infty} \left[(2k+B+3)n + (A-2k-1)\right] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{\mathrm{n}!\, \Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$= \sum_{n=1}^{\infty} \left[(2k+B+3)(n+1) + (A-2k-1)\right] \frac{(\theta)_n \Gamma(\beta) m^{n}}{\mathrm{n}!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$= \sum_{n=1}^{\infty} \left[(2k+B+3)n + (B+A+2)\right] \frac{(\theta)_n \Gamma(\beta) m^{n}}{\mathrm{n}!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$= \left(\frac{2k+B+3}{\alpha}\right) \sum_{n=1}^{\infty} \left[(\alpha n+\beta-1) + (1-\beta)\right] \frac{(\theta)_n \Gamma(\beta) m^{n}}{\mathrm{n}!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$+ (B+A+2) \sum_{n=1}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n}}{\mathrm{n}!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$= \left(\frac{2k+B+3}{\alpha}\right) \sum_{n=1}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n}}{\mathrm{n}!\, \Gamma(\alpha n+\beta-1) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$+ \left[\left(\frac{2k+B+3}{\alpha}\right)(1-\beta) + (B+A+2)\right] \sum_{n=1}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n}}{\mathrm{n}!\, \Gamma(\alpha n+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)}$$

$$= \frac{(\theta)_n \Gamma(\beta)}{n!\, \mathbb{E}_{\alpha,\beta}^{\theta}(m)} \left[\frac{2k+B+3}{\alpha} \left(E_{\alpha,\beta-1}^{\theta}(m) - \frac{1}{\Gamma(\beta-1)}\right)\right.$$

$$\left. + \left[\left(\frac{2k+B+3}{\alpha}\right)(1-\beta) + (B+A+2)\right] \left(E_{\alpha,\beta}^{\theta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)}\right)\right]$$

$$\leq |B-A|,$$

This completes the evidence for Theorem 2.3.

Theorem 2.4. *Let* $\beta > 2$. *Then* $I_{\alpha,\beta}^{m,\theta} \in k - \mathcal{UCV}[A, B]$ *if*

$$\frac{(\theta)_n\Gamma(\beta)}{n!\,\mathbb{E}_{\alpha,\beta}^{\theta}(m)}\Bigg[\frac{2k+B+3}{\alpha^2}\left(E_{\alpha,\beta-2}^{\theta}(m)-\frac{1}{\Gamma(\beta-2)}\right)$$
$$+\left(\frac{(2k+B+3)(3-2\beta)+\alpha(2B+A+2k+5)}{\alpha^2}\right)\left(E_{\alpha,\beta-1}^{\theta}(m)-\frac{1}{\Gamma(\beta-1)}\right)$$
$$+\left(\frac{(2k+B+3)(1-\beta)^2}{\alpha^2}+\frac{(2B+A+2k+5)(1-\beta)}{\alpha}+(B+A+2)\right)\left(E_{\alpha,\beta}^{\theta}(m)-\frac{n!}{(\theta)_n\Gamma(\beta)}\right)\Bigg]$$
$$\leq |B-A|$$

Proof. We consider the same approach of Theorem 2.3 by the means of Lemma 2.2 and (2.2). Here we let

$$J_2 := \sum_{n=2}^{\infty} n[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{(\theta)_n\Gamma(\beta)m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\,\mathbb{E}_{\alpha,\beta}^{\theta}(m)} \leq |B-A|.$$

3. Inclusion Results of $\mathcal{J}_{\alpha,\beta}^{m} f$

Theorem 3.1. *Let* $\beta > 1$. *If* $f \in \mathcal{R}^{\tau}(C, D)$, *then* $\mathcal{J}_{\alpha,\beta}^{m,\theta} f \in k - \mathcal{UCV}[A, B]$ *if*

$$\frac{(C-D)|\tau|(\theta)_n\Gamma(\beta)}{n!\mathbb{E}_{\alpha,\beta}^{\theta}(m)}\left[\frac{2k+B+3}{\alpha}\left(E_{\alpha,\beta-1}(m)-\frac{1}{\Gamma(\beta-1)}\right)\right.$$
$$\left.+\left[\left(\frac{2k+B+3}{\alpha}\right)(1-\beta)+(B+A+2)\right]\left(E_{\alpha,\beta}(m)-\frac{n!}{(\theta)_n\Gamma(\beta)}\right)\right] \tag{3.1}$$
$$\leq \quad |B-A|$$

Proof. Using Lemma 2.2 and (2.1) it is enough to verify that

$$\sum_{n=2}^{\infty} n[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{(\theta)_n\Gamma(\beta)m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}^{\theta}(m)}|a_n| \leq |B-A|$$

Now, since $f \in \mathcal{R}^{\tau}(C, D)$, in view of Lemma 1.1 the coefficients bound is

$$|a_n| \leq \frac{(C-D)|\tau|}{n}, n \in \mathbb{N} \setminus \{1\}$$

Thus, it is sufficient to show that

$$(C-D)|\tau|\left[\sum_{n=2}^{\infty}[2(k+1)(n-1)+|n(B+1)-(A+1)|]\frac{(\theta)_n\Gamma(\beta)m^{n-1}}{n!\,\Gamma(\alpha(n-1)+\beta)\mathbb{E}_{\alpha,\beta}(m)}\right]$$
$$\leq \quad |B-A|.$$

Which is the same approach of the proof of Theorem 2.3, we conclude that $\mathcal{J}_{\alpha,\beta}^{m} f \in k - \mathcal{UCV}[A, B]$ if (3.1) holds true.

4. Inclusion Results of the Integral Operator $\mathcal{G}_{\alpha,\beta}^{m,\theta}$

Following the same previous methods, we can readily deduce the next result

Theorem 4.1. *If $\beta > 1$, the integ iral operator follows*

$$\mathcal{G}_{\alpha,\beta}^{m,\theta}(z) := \int_0^z \frac{I_{\alpha,\beta}^{m,\theta}(t)}{t} dt, z \in \mathbb{U},$$

is in k-$\mathcal{UCV}[A, B]$ if the condition of inequality (2.3) is met.

Proof. By the assumption (1.7) we have

$$\mathcal{G}_{\alpha,\beta}^{m,\theta}(z) = z - \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{(\theta)_n \Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)} \frac{z^n}{n}.$$

Now, using (2.1) and Lemma 2.2, the integral operator; $\mathcal{G}_{\alpha,\beta}^{m}(z)$ belongs to $k - \mathcal{UCV}[A, B]$; if

$$\sum_{n=2}^{\infty} [2(k+1)(n-1) + |n(B+1) - (A+1)|] \frac{(\theta)_n \Gamma(\beta) m^{n-1}}{n! \, \Gamma(\alpha(n-1)+\beta) \mathbb{E}_{\alpha,\beta}^{\theta}(m)} \leq |B - A|$$

we conclude that $\mathcal{G}_{\alpha,\beta}^{m,\theta} \in k - \mathcal{UCV}[A, B]$ if (2.3) holds true.

5. Special Cases

Let $A = 1 - 2\gamma$, and $B = -1$ with $0 \leq \gamma < 1$ in the above theorems, we receive the following special cases:

Corollary 5.1. *Let $\beta > 1$. Then $I_{\alpha,\beta}^{m,\theta} \in k - \mathcal{SD}[k, \gamma]$ if*

$$\frac{(\theta)_n \Gamma(\beta)}{n! \, \mathbb{E}_{\alpha,\beta}^{\theta}(m)} \Big[\frac{k+1}{\alpha} \Big(E_{\alpha,\beta-1}^{\theta}(m) \mathrm{i} - \frac{1}{\Gamma(\beta - 1\mathrm{i})} \Big) + \Big[\Big(\frac{k+1}{\alpha} \Big) \mathrm{i}(1-\beta) + 1 - \gamma \mathrm{i} \Big] \Big(E_{\alpha,\beta}^{\theta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)} \Big) \Big] \leq 1 - \gamma.$$

Corollary 5.2. *Let $\beta > 2$. Then $I_{\alpha,\beta}^{m,\theta} \in k - \mathcal{KD}[k, \gamma]$ if*

$$\frac{(\theta)_n \Gamma(\beta)}{n! \, \mathbb{E}_{\alpha,\beta}(m)} \Big[\frac{k+1}{\alpha^2} \Big(E_{\alpha,\beta-2}^{\theta}(m) - \frac{1}{\Gamma(\beta-2)} \Big) + \Big(\frac{(k+1)(3-2\beta) + \alpha(2-\gamma+k)}{\alpha^2} \mathrm{i} \Big) \Big(E_{\alpha,\beta-1}^{\theta}(m) - \frac{1}{\Gamma(\beta-1)} \Big) + \Big(\frac{(k+1)(1-\beta)^2}{\alpha^2} + \frac{(2-\gamma+k)(1-\beta)}{\alpha} + (1-\alpha) \Big) \Big(E_{\alpha,\beta}^{\theta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)} \Big) \Big] 1 - \gamma$$

Corollary 5.3. *Let $\beta > 1$. If $f \in \mathcal{R}^{\tau}(C, D)$, then $\mathcal{I}_{\alpha,\beta}^{m,\theta} f \in k - \mathcal{KD}[k, \gamma]$ if*

$$\frac{(C-D)|\tau|(\theta)_n \Gamma(\beta)}{n! \, \mathbb{E}_{\alpha,\beta}(m)} \Big[\frac{k+1}{\alpha} \Big(E_{\alpha,\beta-1}^{\theta}(m) - \frac{1}{\Gamma(\beta-1)} \Big) + \Big[\Big(\frac{k+1}{\alpha} \Big) \mathrm{i}(1-\beta) + 1 - \gamma \Big] \Big(E_{\alpha,\beta}^{\theta}(m) - \frac{n!}{(\theta)_n \Gamma(\beta)} \Big) \Big] \leq 1 - \gamma$$

Corollary 5.4. *Let $\beta > 1$. The component operator provided by (4.1) is then in class k; $\mathcal{KD}[k, \gamma]$ if the inequality in Corollary 5.1 holds true.*

6. Conclusions

The generalized Mittag-Leffler function has been investigated by the means of Poisson distribution. A normalized form $\mathbb{E}_{\alpha,\beta}^{\theta}(z)$ has been studied in terms of its inclusion in the well know subclasses of analytic functions, here we have considered $k-\mathcal{ST}[A,B]$ and $k-\mathcal{UCV}[A,B]$. Sufficient conditions are derived for $\mathrm{I}_{\alpha,\beta}^{m,\theta}(z)$, $\mathcal{J}_{\alpha,\beta}^{m}f$ and the integral operator $\mathcal{G}_{\alpha,\beta}^{m,\theta}$ to belong to k-Janowski convex and k-uniformly star-like functions. Lastly, given some A and B parameter values, special cases are discussed.

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Multiplication and Inverse Operations in Parametric Form of Triangular Fuzzy Number

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A B S T R A C T

Many authors have given the arithmetic form of triangular fuzzy numbers, especially for addition and subtraction; however, there is not much difference. The differences occur for multiplication, division, and inverse operations. Several authors define the inverse form of triangular fuzzy numbers in parametric form. However, it always does not obtain $\tilde{a}(r)\otimes\frac{1}{\tilde{a}(r)}\neq\tilde{\imath}(r)$, *because we cannot uniquely determine the inverse that obtains the unique identity. We will not be able to directly determine the inverse of any matrix in the form of a triangular fuzzy number. Thus, all problems using the matrix* $\tilde{A}$ *in the form of a triangular fuzzy number cannot be solved directly by determining* $\tilde{A}^{-1}$. *In addition, there are various authors who, with various methods, try to determine* $\tilde{A}^{-1}$ *but still do not produce* $\tilde{A}\otimes\tilde{A}^{-1}=\tilde{I}$. *Consequently, the solution of a fully fuzzy linear system will produce an incompatible solution, which results in different authors obtaining different solutions for the same fully fuzzy linear system. This paper will promote an alternative method to determine the inverse of a fuzzy triangular number in parametric form. It begins with the construction of a midpoint* $m(\tilde{a})$ *for any triangular fuzzy number* $\tilde{a}=(a,\alpha,\beta)$, *or in parametricform* $\tilde{a}(r)=[\underline{a}(r),\overline{a}(r)]$. *Then the multiplication form will be constructed obtaining a unique inverse which produces* $\tilde{a}(r)\otimes\frac{1}{\tilde{a}(r)}=\tilde{\imath}(r)$. *The multiplication, division,and inverse forms will be proven to satisfy variousalgebraic properties. Therefore, if a triangular fuzzy number is used, and also a triangular fuzzy number matrix is used, it can be easily directly applied to produce a unique inverse. At the end of this paper, we will give an exampleof calculating the inverse of a parametric triangular fuzzy number for various cases. It is expected that the reader can easily develop it in the case of a fuzzy matrix in the form of a triangular fuzzy number.*

Keywords Triangular Fuzzy Number, Multiplication, Inverse in Parametric Form, Triangular Fuzzy Liner System

1. Introduction

Fuzzy linear systems are used in various fields of science,especially engineering, finance, and economics [1-4,8].Some models of fuzzy linear systems include linear systems in the form of triangular fuzzy numbers. The arbitrary triangular fuzzy numbers can be changed in the form of a parametric form triangular fuzzy number, as introduced by some authors, including [5-17,19,23,34].

Some arithmetic forms for triangular fuzzy number operations are introduced by some authors but there is a little difference for addition and subtraction operations.Meanwhile, for multiplication and division/inverseoperations, there are some models. For example,[5,7,17-22] use the concept of min max s for multiplication,but do it differently for the division. On the other hand,[4-6,24-28] provide an alternative to multiplication invarious cases. The author's focus is "why many authors donot provide alternatives for calculating the inverse of a triangular fuzzy number, such as [1,4-7,18-21,25-27]".Furthermore, why is there no author who completestriangular fuzzy number linear system using the concept of determinate or inverse fuzzy matrix? It is suggested that each author looks for an alternative solution and avoid using the inverse of the triangular fuzzy number, even trying to partition it into a real matrix. For example, [25] uses the ST method, while [8] does this by separating the parts $\underline{a}_{ij}(r)$ with $\overline{a}_{ij}(r)$ into separate equations.Furthermore, [23] uses the functions $\underline{f}(\propto)$ and $\overline{f}(\propto)$ and then calculates the limit, while other method used by various authors [1], [5-6,9-16,19,23,29-30] in solving the linear system of fuzzy numbers was either in the basic form or in the form of parametric.

The basics of the problem of various arithmetic operations and various methods of solving the system of linear are given for the arbitrary triangular fuzzy number $\tilde{a}(r)$. There is no element $\frac{1}{\tilde{a}(r)}$, so that $\tilde{a}(r)\otimes\frac{1}{\tilde{a}(r)} = \tilde{I} = [1,0,0]$. Based on the description above, the authors define that the form of the multiplication of two fuzzy numbers for $\tilde{a}(r) \neq \tilde{0}$ will be able to determine a single element $\tilde{x}(r)$, i.e., $\tilde{a}(r)\otimes\tilde{x}(r) = \tilde{x}(r)\otimes\tilde{a}(r) = \tilde{I}$; in this case, means $\tilde{x}(r) = \frac{1}{\tilde{a}(r)}$. Furthermore, the concept of multiplication and inverse can be used easily in solving triangular fuzzy number linear systems and other problems that require the concept of determinant and inverse matrix triangular fuzzy numbers.

2. Preliminaries

Some basic concepts of fuzzy number have been defined in [3,9-16,27].

2.1. Definition

A triangular fuzzy number $\tilde{a} = (a, \alpha, \beta)$ is a fuzzy set on R with the member ship function given which satisfies:

1. $\tilde{a}(x)$ is upper semi-continuous
2. $\tilde{a}(x) =$ 0,outside the interval [0,1]
3. $\tilde{a}(x)$ is a monotonic increasing function on $[a - \alpha, a]$
4. $\tilde{a}(x)$ is a monotonic decreasing function on $[a, a + \beta]$
5. $\tilde{\mu}(x) = 1, for\ x = a$

Notation of the triangular fuzzy number used in this research is $\tilde{a} = (a, \alpha, \beta)$, where a is the center of triangular fuzzy number, α is the distance of left wide, and β is right wide; this notation has been used in [3,9-16,27].The membership function of triangular fuzzy number

$\tilde{a} = (a, \alpha, \beta)$ is:

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 - \frac{a - x}{\propto}, & \text{if } a{-}\propto \le x \le a \\ 1 - \frac{x - a}{\beta}, & \text{if } a \le x \le a + \beta \\ 0, & \text{other} \end{cases}$$

A fuzzy number $\tilde{a}(r)$ in parametric form can be notated as $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ with $\underline{a}(r) = a - (1-r)\alpha$ and $\overline{a}(r) = a + (1-r)\beta$.

2.2. Definition

A fuzzy number $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ is a function which satisfies:

a. $\underline{a}(r)$ is a bounded left continuous non-decreasing function at (0,1], and right continuous at 0,

b. $\overline{a}(r)$ is a bounded left continuous non-increasing function at (0,1], and right continuous at 0,

c. $\underline{a}(r) \le \overline{a}(r), r \in [0,1]$

Two fuzzy numbers $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ and $\tilde{b}(\underline{r}) = [\underline{b}(r), \overline{b}(r)]$ are equal if $\underline{a}(r) = \underline{b}(r)$ and $\overline{a}(r) = b(r)$. The forms $\tilde{a} = (a, \alpha, \beta)$ and $\tilde{b} = (b, \gamma, d)$ are equal if $a = b, \alpha = \gamma$ and $\beta = d$. Definition of similarity between two fuzzy numbers is agreed by some authors. However, there are not many authors who state explicitly about positivity of triangular fuzzy number as [1,4,7,25]. They denote that triangular fuzzy number $\tilde{a} = (a, \alpha, \beta)$ is non-negative if $a \ge 0,$ while [26], denote $\tilde{a}$ is positive and $\tilde{a}$ is negative if $a + \beta < 0.$ Furthermore, algebra of interval in parametric form as given by [1,3,5,7,17,23,34] is as follows.

2.3. Definition

Two fuzzy numbers in parametric form $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ and $\tilde{b}(r) = [\underline{b}(r), \overline{b}(r)]$, and scalar $k \in R$ is defined as follows:

a) $\tilde{a}(r) + \tilde{b}(r) = [\underline{a}(r) + \underline{b}(r), \overline{a}(r) + \overline{b}(r)]$

b) $\tilde{a}(r) - \tilde{b}(r) = [\underline{a}(r) - \overline{b}(r), \overline{a}(r) - \underline{b}(r)]$

c) $\tilde{a}(r) \otimes \tilde{b}(r) = [\min S, \max S]$

   a) With
   $S = \{\underline{a}(r)\underline{b}(r), \underline{a}(r)\overline{b}(r), \overline{a}(r)\underline{b}(r), \overline{a}(r)\overline{b}(r)\}$

d) $k\tilde{a}(r) = \begin{cases} [k\,\overline{a}(r), k\underline{a}(r)], & if\ k < 0 \\ [k\underline{a}(r), k\overline{a}(r)], & if\ k \ge 0 \end{cases}$

e) $m(\tilde{a}) = \frac{\underline{a}(r) + \overline{a}(r)}{2}$

If we apply them to triangular fuzzy number in the form $\tilde{a} = (a, \alpha, \beta)$ and $\tilde{b} = (b, \gamma, d)$, the algebra is given by [1,5,7,17-20,23-25]. If it is changed to parametric form, then the multiplication operation will be the same as Definition 2.3. However, the multiplication operation is different from what is given by [4,7,26-28,32-33], while the concept of positivity of triangular and trapezoidal fuzzy number uses a wide area concept as given in [9-16].

3. Material and Method

As noted above, the addition, subtraction, and scalar multiplication use Definition 2.3 (a), (b), and (d). Meanwhile, the multiplication of fuzzy number will be formulated using another concept. Before formulating the concept of multiplication, we will define the positivity of triangular fuzzy number. In this study, triangular fuzzy numbers are chosen, because the use of triangular fuzzy numbers provides easy and simple calculations. This is because the triangular fuzzy number has the characteristic of its membership function which is linear, although the use of fuzzy numbers with non-triangular membership functions may also be studied.

3.1. Definition

A triangular fuzzy number $\tilde{a} = (a, \alpha, \beta)$ is positive if $a > 0$, and $\tilde{a}$ is negative if $a < 0$. In parametric form, $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)] = [a - (1-r)\alpha, a + (1-r)\beta]$ Define $(\tilde{a}) = \frac{\underline{a}(1)+\overline{a}(1)}{2} = a$, so the positivity concept based on $m(\tilde{a})$ is as follows:

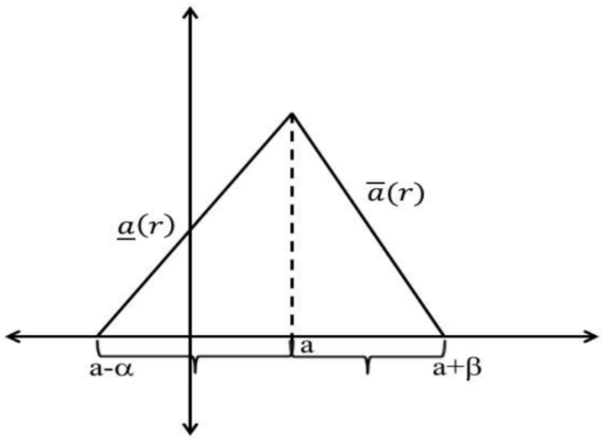

Figure 1. Triangular Fuzzy Number

3.2. Research Model

Parametric triangular fuzzy number $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ is positive if $m(\tilde{a}) > 0$, and it is non-negative if $m(\tilde{a}) \geq 0$. Furthermore, it is negative if $m(\tilde{a}) < 0$, and it is non-positive if $m(\tilde{a}) < 0$, and it is zero if $m(\tilde{a}) = 0$, which is notated with $\tilde{a}(r) \approx \tilde{0}(r)$. $\tilde{a}(r) = [0,0]$ so $\tilde{a}(r)$ is pure zero with the notation $\tilde{a}(r) \approx \tilde{0}_p(r)$.

Both definitions clearly mean that the definition of positivity fuzzy number is equivalent to Definition 3.1 and 3.2. Furthermore, we define the multiplication formula of arbitrary two parametric triangular fuzzy numbers $\tilde{a}(r) = \left[\underline{a}(r), \overline{a}(r)\right]$ and $\tilde{b}(r) = \left[\underline{b}(r), \overline{b}(r)\right]$ are

$$\tilde{a}(r)\otimes\tilde{b}(r) = \left(\underline{a}(r)m\left(\tilde{b}\right) + \underline{b}(r)m(\tilde{a}) - m(\tilde{a})m\left(\tilde{b}\right),\right.$$
$$\left.\overline{a}(r)m\left(\tilde{b}\right) + \overline{b}(r)m(\tilde{a}) - m(\tilde{a}).m\left(\tilde{b}\right)\right) \quad (3.1)$$

Simplifying equation (3.1) to be

$$\tilde{a}(r)\otimes\tilde{b}(r) =$$
$$\left(\underline{a}(r)b + \underline{b}(r)a - ab,\ \overline{a}(r)b + \overline{b}(r)a - ab\right) \quad (3.2)$$

Remark

By definition 3.2 and equation (3.2), we have

(I) If $\tilde{a}(r)$ and $\tilde{b}(r)$ are positive, then $\tilde{a}(r)\otimes\tilde{b}(r)$ positive.

(ii) If $\tilde{a}(r)$ and $\tilde{b}(r)$ are negtive, then $\tilde{a}(r)\otimes\tilde{b}(r)$ positive.

(iii) If $\tilde{a}(r)$ is negative and $\tilde{b}(r)$ is positive, then $\tilde{a}(r)\otimes\tilde{b}(r)$ is negative.

(iv) If $\tilde{a}(r)$ is positive and $\tilde{b}(r)$ is negative, then $\tilde{a}(r)\otimes\tilde{b}(r)$ is negative.

4. Results and Discussion

Based on Definition 3.2 and equation (3.2), we can construct $\frac{1}{\tilde{a}(r)}$ for arbitrary parametric triangular fuzzy number $\tilde{a}(r)$, such as

$$\tilde{a}(r)\otimes\tilde{x}^*(r) = \tilde{I}(r) = [1,1] \quad (4.1)$$

4.1. Theorem

Arbitrary fuzzy number $\tilde{a}(r) = \left[\underline{a}(r), \overline{a}(r)\right]$ where $m(\tilde{a}) \neq 0$, there are

$$\tilde{x}^*(r) = \frac{1}{\tilde{a}(r)} = \left[\frac{2m(\tilde{a}) - \underline{a}(r)}{\left(m(\tilde{a})\right)^2}, \frac{2m(\tilde{a}) - \overline{a}(r)}{\left(m(\tilde{a})\right)^2}\right]$$

then equation (3.1) applies with the multiplication as on equation (3.1) or (3.2).

Proof

First, we determine the value of $m(\tilde{x}^*)$, that is

$$m(\tilde{x}^*) = \underline{x}(1) = \frac{2m(\tilde{a}) - \underline{a}(1)}{\left(m(\tilde{a})\right)^2} = \frac{2m(\tilde{a}) - \overline{a}(1)}{\left(m(\tilde{a})\right)^2}$$

Because the value of $m(\tilde{a}) = \underline{a}(1) = \overline{a}(1) = a$, we get $m(\tilde{x}^*) = \frac{2a-a}{a^2} = \frac{1}{a}$, so that

$$\tilde{a}(r)\otimes\tilde{x}^*(r) = \tilde{I} = [\underline{a}(r)m(\tilde{x}^*) + \underline{x}(r)m(\tilde{a}) - m(\tilde{a})m(\tilde{x}^*), \overline{a}(r)m(\tilde{x}^*) + \overline{x}(r)m(\tilde{a}) - m(\tilde{a}).m(\tilde{x}^*)]$$

$$= \left[\underline{a}(r)\frac{1}{a} + \frac{2m(\tilde{a}) - \underline{a}(r)}{\left(m(\tilde{a})\right)^2}.m(\tilde{a}) - a.\frac{1}{a}, \overline{a}(r)\frac{2}{a} + \frac{2m(\tilde{a}) - \overline{a}(r)}{\left(m(\tilde{a})\right)^2}.m(\tilde{a}) - a.\frac{1}{a}\right]$$

$$= \left[\underline{a}(r)\frac{1}{a} + \frac{2a - \underline{a}(r)}{a} - 1, \overline{a}(r)\frac{1}{a} + \frac{2a - \overline{a}(r)}{a} - 1\right] = [1,1]$$

4.2. Example

An example is given in Table 4.1

Table 4.1. An example of arbitrary parametric triangular fuzzy number $\tilde{a}(r)$

$\tilde{a}(r)$	$\tilde{x}^*(r) = \frac{1}{\tilde{a}(r)}$	$\tilde{a}(r)\otimes\frac{1}{\tilde{a}(r)}$
$(4 + r, 6 - r)$	$\left[\frac{2.5-(4+r)}{25}, \frac{2.5-(6-r)}{25}\right] = \left[\frac{6-r}{25}, \frac{4+r}{25}\right]$	$\left[\left((4 + r).\frac{1}{5} + \frac{6-r}{25}.5\right) - 1, (6 - r).\frac{1}{5} + \frac{4+r}{25}5 - 1\right]$ = [1,1]
(-2+r, 1-2r)	$\frac{2.(-1)-(-2+r)}{1}, \frac{2.(-1)-(1-2r)}{1}$[-r, -3+2r]	[(-2+r)(-1)+(-r)(-1)-1, (1-2r)(-1)+(-3+2r)(-1)-1 = [1,1]
(-2 + 3r, 5-4r)	$[2.1 - (-2 + 3r), 2.1 - (5 - 4r)]$=[4-3r, -3+4r]	[-2+3r).1+(4-3r).1 - 1, (5-4r).1+(-3+4r).1-1]=[1,1]

If $\tilde{b}(r) = [\underline{b}(r), \overline{b}(r)]$ then $m\left(\frac{1}{\tilde{b}(r)}\right) = \left[\frac{2m(\tilde{b})-\underline{b}(r)}{\left(m(\tilde{b})\right)^2}, \frac{2m(\tilde{b})-\overline{b}(r)}{\left(m(\tilde{b})\right)^2}\right] = \frac{1}{m(b)}$

4.3. Corollary

For arbitrary fuzzy numbers $\tilde{a}(r) = [\underline{a}(r), \overline{a}(r)]$ and $\tilde{b}(r) = [\underline{b}(r), \overline{b}(r)]$, we have

$$\frac{\tilde{a}(r)}{\tilde{b}(r)} = \tilde{a}(r)\otimes\frac{1}{\tilde{b}(r)} = [\underline{a}(r), \overline{a}(r)]\otimes\left[\frac{2m(\tilde{b}) - \underline{b}(r)}{\left(m(\tilde{b})\right)^2}, \frac{2m(\tilde{b}) - \overline{b}(r)}{\left(m(\tilde{b})\right)^2}\right]$$

$$= \left[\underline{a}(r)\frac{1}{m(\tilde{b})} + \frac{2m(\tilde{b}) - \underline{b}(r)}{\left(m(\tilde{b})\right)^2}m(\tilde{a}) - \frac{m(\tilde{a})}{m(\tilde{b})}, \quad \overline{a}(r)\frac{1}{m(\tilde{b})} + \frac{2m(\tilde{b}) - \overline{b}(r)}{\left(m(\tilde{b})\right)^2}m(\tilde{a})\right]$$

$$= \left[\frac{\underline{a}(r)m(\tilde{b}) + 2m(\tilde{b})m(\tilde{a}) - \underline{b}(r)m(\tilde{a}) - m(\tilde{a})m(\tilde{b})}{\left(m(\tilde{b})\right)^2}, \frac{\overline{a}(r)m(\tilde{b}) + 2m(\tilde{b})m(\tilde{a}) - \overline{b}(r)m(\tilde{a}) - m(\tilde{a})m(\tilde{b})}{\left(m(\tilde{b})\right)^2}\right]$$

$$= \left[\frac{\underline{a}(r)m(\tilde{b}) - \underline{b}(r)m(\tilde{a}) + m(\tilde{a})m(\tilde{b})}{\left(m(\tilde{b})\right)^2}, \frac{\overline{a}(r)m(\tilde{b}) - \overline{b}(r)m(\tilde{a}) + m(\tilde{a})m(\tilde{b})}{\left(m(\tilde{b})\right)^2}\right]$$

In the same, way as proofs Theorem 4.1 and Corollary 4.3, the following theorem can be proven for arbitrary triangular fuzzy numbers [34-35].

4.4. Theorem

Let $\tilde{a}(r), \tilde{b}(r)$ and $\tilde{c}(r)$ be parametric triangular fuzzy, respectively, we have

a. $\tilde{a}(r)\otimes\tilde{0}(r) = \tilde{0}(r)$

b. $\tilde{a}(r)\otimes\tilde{I}(r) = \tilde{I}(r)$

c. $\tilde{a}(r)\otimes\tilde{b}(r) = \tilde{b}(r)\otimes\tilde{a}(r)$

d. $\left(\tilde{a}(r)\otimes\tilde{b}(r)\right)\otimes\tilde{c}(r) = \tilde{a}(r)\otimes\left(\tilde{b}(r)\otimes\tilde{c}(r)\right)$

e. $\left(\tilde{a}(r)\oplus\tilde{b}(r)\right)\otimes\tilde{c}(r) = \tilde{a}(r)\otimes\tilde{c}(r)\oplus\tilde{b}(r)\otimes\tilde{c}(r)$

f. If $\tilde{a}(r)\otimes\tilde{x}(r) = \tilde{b}(r)$ where $\tilde{a}(r) \not\approx \tilde{0}(r)$, then $\tilde{x}(r) = \frac{\tilde{b}(r)}{\tilde{a}(r)}$

g. If $\tilde{a}(r)\otimes\tilde{b}(r) = \tilde{0}(r)$, then $\tilde{a}(r) \approx \tilde{0}(r)$ or $\tilde{b}(r) \approx \tilde{0}(r)$

h. If $\tilde{a}(r)\otimes\tilde{b}(r) = \tilde{a}(r)\otimes\tilde{c}(r)$ where $\tilde{a}(r) \not\approx \tilde{0}(r)$, then $\tilde{b}(r) = \tilde{c}(r)$

i. If $\tilde{a}(r) \not\approx \tilde{0}(r)$, then $\frac{1}{\tilde{a}(r)} = \tilde{0}(r)$ and $\frac{1}{\frac{1}{\tilde{a}(r)}} = \tilde{a}(r)$

j. If $\tilde{a}(r) \not\approx \tilde{0}(r)$ and $\tilde{b}(r) \neq \tilde{0}(r)$, then $\frac{1}{\tilde{a}(r)\otimes\tilde{b}(r)} = \frac{1}{\tilde{a}(r)}\otimes\frac{1}{\tilde{b}(r)}$

Proof:

Clearly

4.5. Remark

For arbitrary parametric triangular fuzzy number $\tilde{a}(r)$, we have:

$$\frac{\tilde{a}(r)}{\tilde{a}(r)} = \left[\frac{\underline{a}(r)m(\tilde{a}) - \underline{a}(r)m(\tilde{a}) + m(\tilde{a})m(\tilde{a})}{\left(m(\tilde{a})\right)^2}, \frac{\overline{a}(r)m(\tilde{a}) - \overline{a}(r)m(\tilde{a}) + m(\tilde{a})m(\tilde{a})}{\left(m(\tilde{a})\right)^2}\right] = [1,1]$$

5. Conclusion

For arbitrary triangular fuzzy numbers in parametric form $\tilde{a}(r) = \left[\underline{a}(r), \overline{a}(r)\right]$ and $\tilde{b}(r) = \left[\underline{b}(r), \overline{b}(r)\right]$, the addition, subtraction, and scalar multiplication operations are the same as those found by most authors such as the rules (a), (b) and (d) on definition in sub 2.3. Meanwhile, multiplication is as in equation (3.1). For inverse, we used the rule of Theorem on sub 4.1. The rules of algebra operation for this triangular fuzzy number in parametric form can be said to be better than the existing form of operation because its multiplication and division operations are more complete and include wider case.

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A New Methodology on Rough Lattice Using Granular Concepts

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ABSTRACT

Rough set theory has a vital role in the mathematical field of knowledge representation problems. Hence, a Rough algebraic structure is defined by Pawlak. Mathematics and Computer Science have many applications in the field of Lattice. The principle of the ordered set has been analyzed in logic programming for crypto-protocols. Iwinski extended an approach towards the lattice set with the rough set theory whereas an algebraic structure based on a rough lattice depends on indiscernibility relation which was established by Chakraborty. Granular means piecewise knowledge, grouping with similar elements. The universe set was partitioned by an indiscernibility relation to form a Granular. This structure was framed to describe the Rough set theory and to study its corresponding Rough approximation space. Analysis of the reduction of granular from the information table is based on object-oriented. An ordered pair of distributive lattices emphasize the congruence class to define its projection. This projection of distributive lattice is analyzed by a lemma defin ing that the largest and the smallest elements are trivial ordered sets of an index. A Rough approximation space was examined to incorporate with the upper approximation and analysis with various possibilities. The Cartesian product of the lattice was investigated. A Lattice homomorphism was exam ined with an equivalence relation and its conditions. Hence the approximation space exists in its union and intersection in the upper approximation. The lower approximation in different subsets of the distributive lattice was studied. The generalized lower and upper approximations were established to verify some of the results and their properties.

Keywords Indiscernibility Relation, Granular Lattice,Congruence Class, Distributive Lattice, Lattice Homomorphism Figure

1 Introduction

Rough Set Theory was introduced by Pawlak [1] for a study of vague and uncertain data with complete and incomplete knowledge. An approximation space was farmed to be defined by a universal set and equivalence relation. The methodology was studied as a pair of subsets namely lower and upper ap proximation. An abstract approximation space was introduced by Cattaneo [2] and the relation between the orthocomplement operates was studied. A wide investigation was conducted to the model and its approaches toward the Rough Set Theory. Yao [3] studied the binary relation between two universal sets as objects and verifies its properties. Emphasis was put on the features of the concept Lattice with Rough

Set Theory. Yao [3] studied the binary relation between two universal sets as objects and verifies its properties. Emphasis was put on the features of the concept Lattice with Rough Set Theory. Susanta Bera [4] discussed the properties of Rough Lattice and Rough Modular Lattice. An approximation space emphasizes the indiscernibility relation of an object by Pawlak notations.

Richard [5] defined an algebraic structure in a congruence class under the closed operation as meet. The properties of congruence classes were applied in translation by Lattice. A topological space for mapping a homomorphic condition was intro duced and its uniqueness theorem was established. Complete Boolean algebra in a new methodology was constructed and its characteristics space in an approximation sense was discussed. Jarvinen [6] examined the different types of lattice and their complement conditions. The operators were discussed with various approximation spaces to define a Rough Set Estaji [7], an algebraic structure was defined to describe an in terconnection between the concept of Rough set and Lattices, and the properties of Rough Ideal and Rough Filter were dis cussed. A homomorphism function was described for a Prime ideal and a Prime filter for a set of fixed points. Fei Li [8] investigated Rough groups, and Rough Quotient groups and examined their results with some of their properties. Shao [9] an ordered structure was emphasized with the binary relation on Rough Set Theory, which implies the Lattice Theory. The rough lattice with a specified notation for lower and upper ap proximation was studied.

Yamaguchi [10] introduced a Grey Rough Set using Lattice operation. An information system based on numerical interval data was investigated. An equivalence relation was defined for a grey rough set based on the Lattice operation. A Method ology was introduced for a non-deterministic Information sys tem. Rana [11] approached significant results on the model based on a Rough interval using Lattice to define operators. A distributive lattice has been examined by a family of Rough intervals. An effective algebraic structure has been investigated in the data for some types of Rough Sets [12]. This illustrates a rough approximation space with the covering of the lattice. Rana [13] formulated the two important concepts of Rough Partial ordered relation with Rough Lattice. Rough Boolean Lattice was constructed to verify its properties with Rough relation and Rough Lattice. Yao [14] constructed an approximation op erator using the concept of Lattice. The data were examined by defining a binary relation using the concept of Lattice. An universe set was partitioned by the nested granular to define an equivalence relation [15]. This illustrates the Rough set approximation for a different level of bounded lattice to examine its results and properties. From the above literature, a distributive lattice is framed to de fine a Lattice homomorphism function and verify its equiva lency condition. The Cartesian product of distributive lattice was partitioned by congruence classes and its properties with granular Lattice were discussed. Hence the results and properties were examined.

2 Preliminaries

By partitioning the distributive Lattice, an algebraic struc ture was defined for a lower and upper approximation.

Throughout this paper, "$\leq$" represents the order of a given Lattice.

2.1 Rough Set

An information system (IS) was introduced by Pawlak [1] which consists of a non-empty finite set of an object (O), knowledge obtained as an attribute (A) where A is a combination of condition attributes (C) and decision attributes (D) such as $A = C \cup D$, V is a cartesian product of object and attributes ($O \times A$) and f is a function defined as $f : A \rightarrow V$. Hence, the information system is denoted as $IS = \langle O, A, V, f \rangle$. Let S be a subset of A, then an indiscernibility relation was described as an approximation space is defined as a pair of (O, [p]S) where [p]S is an equivalence relation which partitionthe versal set O. Consider X be a subset of object (O), then the lower and upper approximation is defined as [7, 9]

$$\overline{apr(X)} = \{p \in O | [p]S \subseteq X\}$$

$\underline{apr(X)} = \{p \in O | [p]S \cap X \neq \emptyset\}$

The Boundary, Positive and Negative regions are

1. $BR(X) = \overline{apr(X)} - \underline{apr(X)}$
2. $PR(X) = \underline{apr(X)}$
3. $NR(X) = U - \overline{apr(X)}$

The set is said to be a definable set if $\overline{apr(X)} = \underline{apr(X)}$, else the set is Rough Set or undefinable set.

2.2 Lattice

A partially ordered set $(L, \leq)$ is said to be a Lattice if itsatisfies the condition $p \wedge q = p$ and $p \vee q = q$ for all $p, q \in L$.Here $\wedge$ and $\vee$ represent the binary operators, where $p \leq q$.[5]

2.2.1 Distributive lattice

A distributive Lattice is a Lattice with the operator $\wedge$ and $\vee$ if it satisfies $\forall p, q, r \in L$. [5]

$$\mathbf{p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)}$$
$$\mathbf{p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)}$$

2.3 Rough Lattice

Consider the approximation space (L, P) where P is an equivalence relation. Let $X \subseteq L$ and then the Rough Set was defined as pair of $P(X) = (\underline{P(X)}, \overline{P(X)})$. [6, 10]

Consider a Rough lattice $\langle \overline{P(X)}, \wedge, \vee \rangle$ where $\overline{P(X)}$ are sublattice of L in which it satisfies the following condition for $p, q, r \in X$

1. $p \wedge p = p, p \vee p = p$ (Idempotency)
2. $p \wedge q = q \wedge p, p \vee q = q \vee p$ (Commutativity)
3. $p \wedge (q \wedge r) = (p \wedge q) \wedge r, p \vee (q \vee r) = (p \vee q) \vee r$ (Associativity)
4. $p \wedge (p \vee q) = p, p \vee (p \wedge q) = p$ (Absorption)

5. $p \leq q$ *if* $p \wedge q = p$ *and* $p \vee q = q$ (Consistency).

3 Main Results

This section analyzes the information table into granules and describes the Cartesian product of two distributive lattices. An equivalence class was determined by the lattice homomor phism. A Granular Distributive Lattice with some of the results and properties are introduced.

Definition 3.1. *Consider an approximation space (A, O, Ind) where A is an attribute, O is an object, and* $Ind \subseteq A \times O$ *is an indiscernibility relation between A and O.*

Definition 3.2. *Let* $X \subseteq O$ *and* $M \subseteq A$ *be the dual operator such that* [9, 13]

$M^* = \{x | x \in O, a \in M, (a, x) \in Ind\}$

$X^* = \{a | a \in M, \forall x \in O, (a, x) \in Ind\}$

Here M is the family of all objects that are paired with all the attributes M and X* is the family of all attributes that are fulfilled with all the objects in X.*

Definition 3.3. *A pair of (M, X) is said to be a granular where M A and X O. If X* = M and X = M* , then (M1, X1) ≤ (M2, X2) which implies M1 M2.*

Definition 3.4. *Let (M1, X1) and (M2, X2) be the two granular if X1 = X2 and M1 ≤ M2 then L(M2, X2, G) be the Lattice (L) and induced by the granular (G).*

Definition 3.5. *Consider L(M1, X1, G1) and L(M2, X2, G2)be the granular. Hence the least upper bound and greatest lower bound of the lattice are defined as [8, 10] (M1, X1) (M2, X2) = (M1 ∩ M2,(X1 X2)**)(M1, X1) (M2, X2) = ((M1 M2)**, X1 ∩ X2)*

Definition 3.6. *Let L(M1, X1, G1) and L(M2, X2, G2) be the two granular in the Lattice then (M, X) L(M1, X1, G1) then there exist (M' , X') L(M1, X1, G1) such that M' = M then L(M1, X1, G1) ≤ L(M2, X2, G2) [6] .*

Definition 3.7. *Consider a distributive Lattice from a Granular as {Dα|α I} where I be an index set of the partially ordered if (i) Dα = L (ii) If Dα ≤ Dβ then x Dα and y Dβ there exist a mapping π : Dα × Dβ → L such that [x, y]π x ≡ y(L) if (x y') (x' y) L Hence for a canonical structure of the distributive lattice, a congruence relation is defined as [x, y]π.*

Definition 3.8. *Consider M be a sublattice of L then an ap proximation space is defined as (L, M, π). Hence the lower and upper approximation as π(M) = {a [x, y]π|π(a) M} π(M) = {a [x, y]π|π(a) ∩ M ≠ } BRπ(M) = {π(M) − π(M)} A pair of (π(M), π(M)) be the Rough Set on M.*

Theorem 3.1. *Let π be an equivalence relation on an approx imation space (L, M, π) if and only if it satisfies the Lattice homomorphism.*

Proof. *Consider π be an Lattice homomorphism as π : Dα ×Dβ → L then there exist [x, y]π x ≡ y(L) if (x y*

Dβ → L then

there exist [x, y]π x ≡ y(L) if (x y') (x' y) L such that π(x) = a x and π(y) = a y for all x Dα, y Dβ and a is a distributive element in DαorDβ.

Here, π is an binary relation on L. Then xπy π(x) = π(y) Therefore π is an equivalence relation on approximation space (L, M, π).

Similarly, converse is also true.

Theorem 3.2. *Let (L, M, π) be an approximation space then there exist any two elements in L such that its union exists in upper approximation.*

Proof. *Consider Dα ≤ Dβ then there exist x Dα and y Dβ By Lattice homomorphism, π(x), π(y) π(M) Therefore, by distributive lattice if x Dα and y Dbeta then π(x y) = π(x) π(y) π(M) for all x y L Hence the result.*

Corollary 3.3. *Let (L, M, π) be an approximation space then there exist any two elements in L such that its intersection exists in upper approximation.*

Proof. *Hence the proof is similar to Theorem 3.2*

Properties 3.1. *Let (L, M, π) be an approximation space then the following properties holds:*

(I) $\underline{\pi(M)}$ M $\overline{\pi(M)}$

(ii) $\underline{\pi(M\ \ N)}$ $\underline{\pi(M)}$ $\underline{\pi(N)}$

(iii) $\overline{\pi(M\ \ N)}$ $\overline{\pi(M)}$ $\overline{\pi(N)}$

(iv) $\underline{\pi(M \cap N)}$ $\underline{\pi(M)}$ $\underline{\pi(N)}$

(v) $\overline{\pi(M \cap N)}$ $\overline{\pi(M)}$ $\overline{\pi(N)}$

(vi) If M N then π(M) π(N) and $\underline{\pi(M)}$ $\underline{\pi(N)}$

Example 3.1. *Consider an information table with the object of 5 with the attribute price = P, Room = R, F urniture = F, also with decision attributes P restigiousflat = P F consisting of "Yes" and "No".*

Object/Attributes	*Price*	*Room*	*Furniture*	*PrestigiousFlat*
1	*High*	1	*No*	*Yes*
2	*High*	3	*Yes*	*Yes*
3	*High*	1	*No*	*No*
4	*High*	2	*No*	*Yes*
5	*Low*	2	*Yes*	*No*

Table 1. Infomation Table

By partitioning the set as (M, X, G) with the concepts of Granule we can define 23 Granule sets. Hence, reduction of the Granule concept can reduce 9 Granule and form the Hasse Diagram shown below

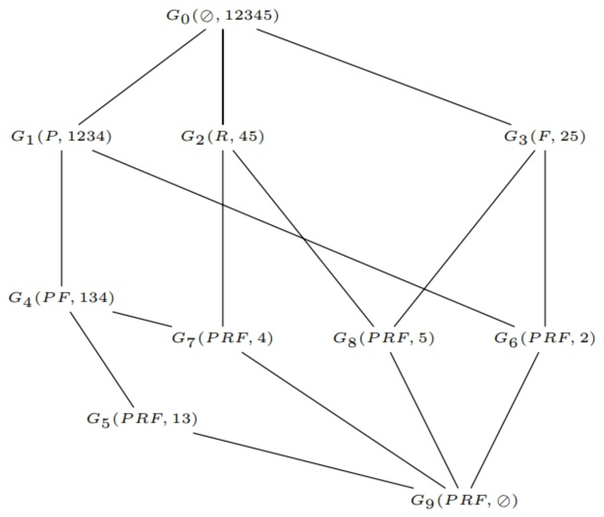

Figure 1. Hasse Diagram

Let us consider Dα = {G0, G1, G4, G5, G9} and Dβ = {G0, G1, G4, G5, G6, G7, G9}

Then the Cartesian product of Dα × Dβ = {(G0, G0),(G0, G1),(G0, G4),(G0, G5),(G0, G6),(G0, G7),(G0, G9),(G1, G0),(G1, G1),(G1, G4),(G1, G5),(G1, G6),(G1, G7),(G1, G9),(G4, G0),(G4, G1),(G4, G4),(G4, G5),(G4, G6),(G4, G7),(G4, G9),(G5, G0),(G5, G1),(G5, G4),(G5, G5),(G5, G6),(G5, G7),(G5, G9),(G9, G0),(G9, G1),(G9, G4),(G9, G5),(G9, G6),(G9, G7),(G9, G9)}

Therefore, [x, y]π = {{G0, G1, G4, G9}, {G0, G1, G4, G9},{G0, G1, G4, G5, G6, G7, G9}, {G0, G1, G4, G5, G7, G9}}

Case (i) M = {G0, G2, G8, G9}

Then, π(M) =   and π(M) = {G0, G1, G4, G5, G6, G7, G9}

Case (ii) M = {G0, G1, G6, G9}

Then, π(M) = {G0, G1, G6, G9}and π(M) = {G0, G1, G4, G5, G6, G7, G9}

Lemma 3.4. *Let a distributive lattice {Dα|α   I} have the largest and smallest element then I be trivially ordered if the same holds with no hypothesis on Dα.*

Proof. *Let us consider 0 or   and 1 or O(Object) be the smallest and largest element in Dα where α   I respectively, then By the definition of homomorphism function π : Dα×Dβ → L, then π be a binary relation. Hence π(x) = π(y)*

π(x) = 1 if α > β

π(x) = x if α = β

π(x) = 0 if α   β

In case of I is trivially ordered, there exist a   Dα such that

Theorem 3.5. *The Cartesian product of a family of {Dα|α I} be a distributive lattice if and only if Dα is a projection for each α I.*

Proof. *Let Dα and Dβ be the distributive Lattice, where α, β I*
By the definition of homomorphism function π : Dα×Dβ → L, then π be a binary relation. Hence π(x) = π(y) From the above Lemma, Dα is projection, there exist an π(x) = π(y) for all x Dα, y Dβ.
Hence the result.

Theorem 3.6. *Let (L, M, π) be an approximation space then Case (i) If M is not a subset of Dβ then the lower approximation is a empty set. Case (ii) If M is a subset of Dβ then thelower approximation is a non-empty set. Case (iii) Always the upper approximation be the Dβ*

Proof. *Let (L, M, π) be the approximation Space then Case (I): Let M is not a subset of Dβ. i.e., M Dβ Hence, y Dβ which implies y / M Since y π(a) M then y / π(a), lower approximation is an empty set.*
Case (ii): Let M is a subset of Dβ. i.e., M Dβ Hence, y Dβ which implies y M Since y π(a) M then y M and y π then lower approximation is always non-empty set.
Case (iii): For any x Dα and y Dβ then, By definition of upper approximation, Dβ π(a) ∩ M which implies Dβ π(M) ——(1) [x, y]π Dα × Dβ then there exist a [x, y]π such that [x, y]π π(a) ∩ M Since its satisfies the distributive condition, π(a) ∩ M Dβ which implies π(M) Dβ ——(2)
Consider *Dα{G0, G1, G5, G10, G15}* and *Dβ{G0, G1, G5, G6, G10, G13, G15}* Hence the ordered pair of distributive lattice as *Dα × Dβ = {(G0, G0),(G0, G1),(G0, G5),(G0, G6),(G0, G10),(G0, G13),(G0, G15),(G1, G0),(G1, G1),(G1, G5),(G1, G6),(G1, G10),(G1, G13),(G1, G15),(G5, G0),(G5, G1),(G5, G5),(G5, G6),(G5, G10),(G5, G13),(G5, G15),(G10, G0),(G10, G1),(G10, G5),(G10, G6),(G10, G10),(G10, G13),(G10, G15),(G15, G0),(G15, G1),(G15, G5),(G15, G6),(G15, G10),(G15, G13),(G15, G15)}*

From Definition 3.7, the equivalence class is defined as *[x, y]π ⇒ x ≡ y(L) if (x∧y′)∨(x′∧y)* ∈ L and From theorem 3.1, it can be verified with Lattice homomorphism condition below.
[G0, G0]π = (123456∧ ∅)∨(∅ ∧123456) = ∅ = G15 ∈ L [G0, G1]π = (123456 ∧ 256) ∨ (∅ ∧ 134) = 256 = G7 ∈ L *[G0, G5]π = (123456 ∧ 2356) ∨ (∅ ∧ 14) = 2356 ∈/ L*
Similarly, find for all the ordered pair of the distributive lattice. Hence the equivalence class is defined as *{{G0, G1, G10, G13, G15}, {G0, G1, G5, G6, G10, G13, G15}, {G1, G5, G6, G10, G13, G15}}* and also, consider *M = {G0, G1, G5, G10, G13, G15}* From this lower and the upper approximation as *π(M) = {G0, G1, G10, G13, G15} π(M) = {G0, G1, G5, G6, G10, G13, G15}*

From the upper approximation G10 and G13 contain the object which satisfies all the attributes and also it contains in *G0, G1, G5, G6*. Hence, we concluded that the object 1 and 4 can shift to the general ward from the ICU ward.

It is observed that the 6 patients data were analyzed in the ICU ward and an ordered pair of granular exists to form a distributive lattice using a Hasse diagram. The Cartesian product of two distributive lattices was given and the equivalence class was verified by the lattice homomorphism condition. Consider $M \subseteq L$, then based on M it is easy to diagonalize the patient shift to the general ward or needs more observation regarding the patient.

5 Conclusions

In this paper,

- A lattice homomorphism was discussed to define an equivalence relation.
- The union and intersection of the upper approximation were discussed based on the structure of the distributive lattice.
- The projection of the distributive lattice was verified with the lemma.
- Always the upper approximation is the sub lattice and the lower approximation differed from the case.
- A Real time experimental analysis uses Granular distributive lattice.

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Even Vertex ζ-Graceful Labeling on Rough Graph

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A B S T R A C T

The study of set of objects with imprecise knowledge and vague information is known as rough set theory. The diagrammatic representation of this type of information may be handled through graphs for better decision making. Tong He and K. Shi introduced the constructional processes of rough graph in 2006 followed by the notion of edge rough graph.They constructed rough graph through set approximations called upper and lower approximations. He et al developed the concept of weighted rough graph with weighted attributes.Labelling is the process of making the graph into a more sensible way. In this process, integers are assigned for vertices of a graph so that we will be getting distinct weights for edges. Weight of an edge brings the degree of relationship between vertices. In this paper we have considered the rough graph constructed through rough membership values and as well as envisaged a novel type of labeling called Even vertex ζ-graceful labeling as weight value for edges. In case of rough graph, weight of an edge will identify the consistent attribute even though the information system is imprecise. We have investigated this labeling for some special graphs like rough path graph, rough cycle graph, rough comb graph, rough ladder graph and rough star graph etc. This Even vertex ζ-graceful labeling will be useful in feature extraction process and it leads to graph mining.

Keywords Rough Graph, Rough Path Graph, Rough Cycle Graph, Rough Comb Graph, Rough Ladder Graph, Rough Star Graph

1 Introduction

Rough set theory proposed by Pawlak in 1982 is a novel mathematical tool for solving uncertain problems through in discernibility relation between the objects. The core premise of Rough set theory is based on lower and upper approxi mations [3]. One of the major important concepts in Rough Set is rough membership function [4] which has a wide range of application in the field of knowledge discovery, data mining,image processing, conflict analysis, decision making processes etc. He T and Shi K introduced rough graph using bi nary relations and its structure [5]. T He, Y Chen, and K Shi first established weighted rough graph in 2006 [6] using the class weights for the edge equivalence class and also appli cation in relationship analysis is explained to show the effec tiveness of generalized Kruskal algorithm to explore the class optimal tree. In 2011, M. Liang, B. Liang, L. Wei and X. Xu defined edge rough graph based on the edge set in which it is said that any pair of graphs can be approximated by a cay ley group. At last, to compute the clique number of groups, it was proved that how the edge rough graph is applied [22]. In 2012, Tong he described the structure of rough graph in three representation forms such as rough figure, edge adjacency ma trix, edge list [7] and properties of rough graph are discussed through the concept of edge precision, rough equal and rough similarity degree which are used to compare different rough graphs [21]. Following them Chen, Jinkun and Jinjin Li de scribed an application of rough sets to graph theory [8]. In this paper, the concept of quasi-outer definable sets has been intro duced and an algorithm is designed for testing bipartiteness of a simple undirected graph [8]. Chellathurai and Jesmalar defined a weighted rough graph and its properties [9]. A vertex rough graph was recently constructed by Bibin Mathew et al. based on indiscernibility relation on vertex set. Vertex precision and edge precision were also defined and discussed in this paper.

He used the rough membership function to demonstrate the rough vertex similarity degree, rough edge similarity degree, and rough equal. Additionally, rough vertex membership and rough edge membership functions were developed with some properties [20]. Rough graphs with additional metrics based on neighbourhood system and its mathematical properties were introduced by Anitha and Arunadevi [19].

Labeling of a graph G is an assignment of integers to the vertices of G or edges of G or both satisfying certain condi tions. A survey of graph labeling is compiled by Gallian [2]. In 1967, Rosa initially proposed graceful labeling of graphG under the name β-valuation, which Solomon W. Golomb later on adopted. The resulting edges are different when each edge e = uv is assigned as graceful labeling $|f(u) - f(v)|$, which is an injection f from the set of vertices V(G) to the set $\{0, 1, 2, \ldots, q\}$. A graph which admits a graceful labeling is called a graceful graph [2]. In 1991, Gnanajothi introduced a labeling of G called odd graceful labeling [2] and in 1985, Lo introduced a labeling of G called edge graceful labeling [2]. Edge odd graceful labeling is a labeling of G that Solairaju and Chithra introduced in 2009 [2]. Likewise, S.N. Daoud demon strated necessary and sufficient conditions of some of the path graphs and cycle related graphs including friendship, wheel, helm, double wheel, gear and fan graph are edge odd graceful labeled in 2017 [11]. Solairaju et al. defined graceful labeling for ladder, sun extension graph, double fan and open staircase graph [12]. Here the definition is given that $C9 * (K1 + Cn)$ is connected whose vertex set is $\{v1, v2, \ldots vn+9\}$ and edge set is $\{vi, vi+1; i = 1 \text{ to } (n - 1)\} \cup \{vn+1, vi; i = 1 \text{ ton}\} \cup \{vi, vi+1; i = n+1, n+2, \ldots n+8\} \cup \{vn+9, vn+1\}$ and it is proved as edge odd graceful [12]. Mohamed R. Zeen El Deen introduced Edge δ-graceful labeling for some cyclic related graphs [13] and also he proved some results in edge even graceful labeling of the join of two graphs [14]. Md Forhad Hossain et al. discussed new classes of graceful trees [18] and Md Shahbaz Aasai and Md Asif et al. computed radio number and radio mean number of lexicographic product of some graphs namely $P2\ [Pq]$, $P3\ [Pq]$, $P2\ [Cq]$ and $P3\ [Cq]$ for $q \geq 5$ and also composed computer code using python lan guage. Hennig Fernaua et al. constructed a sum labeling for fq,p and the proof is given for fq,p is 2-optimal summable and it is proved that except for $f4,p$ and $f5,p$ are sum labelingfor all flowers [15]. Abdullah Zhraaa and Arif Nabeel et al. in troduced dividing graceful labelling for certain types of binary trees, path, caterpillar, star $DS1,k$ and spider [16]. In this paper, we introduce a new type of labeling based on graceful and we investigate it in some rough graphs.

2 Materials and Methods

2.1 Information Table [4]

An information system is a pair IS=(U,A) where U is a non empty ,finite set called the universe and Ais a non empty, finite set of attributes, $a : U \rightarrow Va \text{ for } a \in A$, Va is called the value set of a.

2.2 Indiscernibility [4]

Indiscernibility relation is a central concept in rough set theory. Given a subset of attributes $B \subseteq R$, an indiscernibility relation $ind(B)$ on the universe U can be defined as follows:

$IND(B) = \{(x, y) \in U * U; \forall a \in B, a(x) = a(y)\}$. Indiscernibility relation is an equivalence relation where all identical objects of set are considered as elementary.

2.3 Approximation of Sets [4]

If $M = (U, K)$ is an information system, $F \subseteq K$ and $X \subseteq U$ then the sets $FX = \{x \in U : [x]F \subseteq X\}$ and $FX = \{x \in X : [x]F \cap X \neq$ are called the F-lower and the F-upper approximation of X in K.

2.4 Rough Membership Function [10]

Assume $M = (U, K)$ is an information system, a non empty set F $\subseteq$ U. In rough terms, here is the membership function for the set $\omega_X^F = \frac{|[x]_F \cap F|}{|[x]_F|}$ for some $x \in U$.

2.5 Properties of Rough Membership Function [4]

Some properties of rough membership function as follows:

1. $\mu_X^K(x) = 1 \text{ iff } x \quad K(X)$
2. $\mu_X^K(x) = 0 \text{ iff } x \quad U - KX$
3. $0 < \mu_X^K(x) < 1 \text{ iff } x \quad BNK(X)$
4. If $IND(K) = \{(x, x) : x \quad U\}$ then $\mu_X^K(x)$ *is the characteristic function of X.*
5. If $xIND(K)y$ *then* $\mu KX(x) = \mu KX(y)$
6. $\mu_X^K - X(x) = 1 - \mu_X^K(x)$ *for any* $x \quad X$.
7. $\mu_{X \cap Y}^K(x) \geq max(\mu \quad {}_X^K\cdot), \mu \quad {}_X^K\cdot))$ *for any* $x \quad U$.
8. $\mu_{X \cap Y}^K(x) \leq min\left(ii_X^K(x), \mu_Y^K(x)\right)$ for any $x \in U$
9. $\mu_{\cup X}^K(x) = \sum_{x \in X} \mu_X^K(x)$

2.6 Rough Graph [10]

Let $U = \{V, E, \omega\}$ be a triple consisting of non empty set $V = \{v1, v2, \ldots .vn\} = U$, where U is a universe, $E = \{e1, e2, \ldots ..en\}$ be a set of unordered pairs of distinct elements of V and ω be a function $\omega : V \rightarrow [0,$

1] . A rough graph is defined as

$$\mathbb{U}(v_i, v_j) = \begin{cases} max\left(\omega_G^V(v_i), \omega_G^V(v_j)\right) > 0, \text{edge} \\ max\left(\omega_G^V(v_i), \omega_G^V(v_j)\right) = 0, \text{ no edge.} \end{cases}$$

2.7 Example

Let *U* be an information system. *Let U = {a, b, c, d, e, f, g, h, i}* and the set *X = {a, b, c}*. Let blood pressure, hypertension and complication are condi tion attributes and delivery be the decision attribute. The information table is given as follows:

OBJECTS	C1	C2	C3	D
a	0	1	0	1
b	0	1	0	1
c	0	1	1	1
d	0	0	1	2
e	0	0	1	2
f	1	0	1	2
g	1	0	0	3
h	1	0	0	3
i	1	1	0	3

R{a} = {a, b} = R{b}, R{c} = {c}, R{d}={d, e}=R{e},

R{f} = {f}, R{g} = {g, h} = R{h}, R{i} = {i}

Rough Membership values are

$$\omega_G(\text{a}) = \frac{|\{a,b\} \cap \{a,b,c\}|}{|\{a,b\}|} = 1$$

$$\omega_G(b) = 1, \omega_G(c) = 1, \omega_G(d) = 0, \omega_G(e) = 0, \omega_G(f) = 0,$$
$$\omega_G(g) = 0, \ \ \omega_G(h) = 0, \ \ \omega_G(i) = 0.$$

The Rough graph is constructed for the above information table:

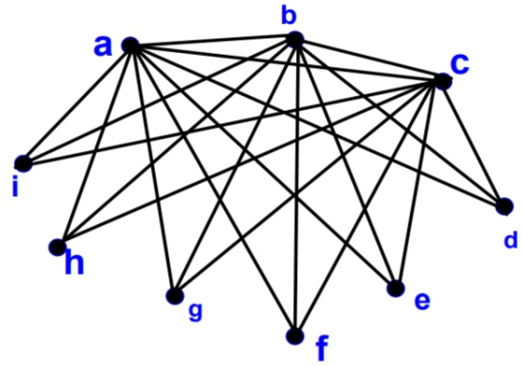

Figure 1. Rough graph.

2.8 Rough Path Graph [10]

Distinct edges in Rough walk are said to be rough trail and distinct vertices in a rough walk are said to be rough path. It is denoted by *Pn*.

2.9 Rough Path Graph [10]

A Rough cycile is defiend as the closed Rough walk *v1, v2, . . . vn = v* where *n ≥ 3* and *v1, v2, . . . v(n−1)* are distinct. It is denoted by *Cn.*

2.10 Rough Ladder Graph [10]

The Ladder Rough graph is defined as the Rough Cartesian product of Rough path and the Complete Rough graph. It is denoted by *Ln.*

3 Main Results

3.1 Even Vertex ζ-Graceful Labeling

A function is called even vertex ζ-graceful labeling of a graph *G(V, E)* with n vertices and m edges if *f : V (G) → {2, 4, 6 . . .}* is bijection and the induced function *f : E (G) →*
N is distinct and *m (G)* = no. of edges then it is defined as

$$f^*(uv) = \begin{cases} \frac{\zeta}{2} & when\ \zeta\ is\ even \\ \frac{\zeta+1}{2} & when\ \zeta\ is\ odd \end{cases} \quad where\ \zeta = f(u) + f(v) + m(G)\ \textbf{for all u,v} \in E\ are\ all\ distinct.$$

3.2 Rough ζ-Graceful Graph

A rough graph *Rφ (G) = (Vφ, Eφ, ω)* has n vertices and m edges if *V φ = {vφ1, vφ2, . . . vφn }, σ : E (G) →* N and *ωφ :Vφ * Vφ →* [0, 1] is bijection such that the labeling of verticesand edges is distinct. Then *Rφ (G) = (Vφ, Eφ, ωφ)* is calledrough ζ-labeling graph if it satisfies the following conditions:

1. If max $(\omega(v_i^{\varphi}),\omega(v_j^{\varphi})) > 0,$ edge exists .
2. If $\sigma^{\varphi}(uv) = \frac{\zeta}{2},\ \zeta\ is\ even\ and\ \sigma^{\varphi}(uv) = \frac{\zeta+1}{2},\ \zeta\ is\ odd\ \textbf{for all u,v} \in V\ where\ \zeta = f(u) f(v) + m(G)\ \textbf{for all u,v} \in V.$

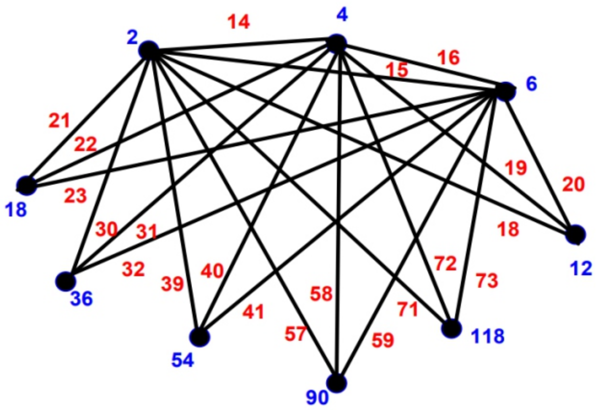

Figure 2. Even vertex ζ-graceful rough graph.

4 Even Vertex ζ-Graceful Labeling for Some Simple Rough Graphs

4.1 Theorem

The rough path graph admits even vertex ζ-graceful labeling for all $n \geq 2$.

Proof: Let Pn be a rough path graph with n vertices and $n - 1$ edges which represents in Figure 3.

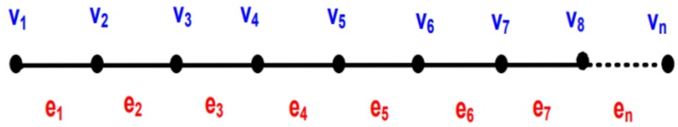

Figure 3. Rough path *Pn.*

Defining the vertex label with the function if $f : V(G) \rightarrow \{2, 4, 6 \ldots\}$ by $f(v_i) = 2i$ for $1 \leq i \leq n$. The edge labels are defined into two cases:

Case (I): If n is even then the mapping for the edge labeling is defined as $f : E(G) \rightarrow N$ with the function $f(ei) = \dfrac{n+2+4i}{2}$ for $i = 1, 2, 3 \ldots n-1$.

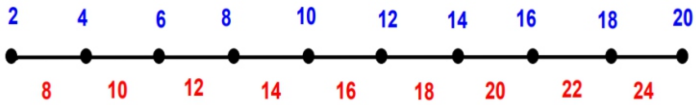

Figure 4. Rough path P10.

Case(ii): If n is odd then the edge labeling is defined as follows: $f(e_i) = \dfrac{n+1+4i}{2}$ for $i = 1, 2, \ldots n-1$.

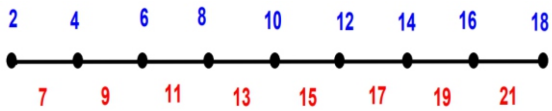

Figure 5. Rough path *P9.*

4.2 Theorem

The rough cycle C_n admits even vertex ζ-graceful labeling for all $n \geq 3$.

Proof: Let $v1, v2, \ldots, vn$ be the vertices of Cn and $e1, e2, \ldots, en$ be the edges of Cn. There are two cases:

Case(I): If n is odd then the vertex labeling is defined as $f(vi) = 2i$ for $1 \le i \le n$. The induced edge labels are as follows:

$$f(e_i) = \frac{n+6+j}{2} \text{ where } \begin{cases} j = 4i - 3 & for\ i = 1, 2, \ldots n - 1 \\ j = 2i - 3 & for\ i = n \end{cases}$$

Case(ii): If n is even then we define label of the vertices of Cn as follows:

$f(vi) = 2i,\ 1 \le i \le n - 1, f(vi) = 2i + 2,\ i = n.$

There exists the induced edge labels as follows:

$$f(e_i) = \frac{n+5+j}{2} \text{ where } \begin{cases} j = 4i - 3, & i = 1, 2, \ldots n - 2 \\ j = 4i - 1, & i = n - 1 \\ j = 2i - 1, & i = n \end{cases}$$

Hence the edges are distinct.

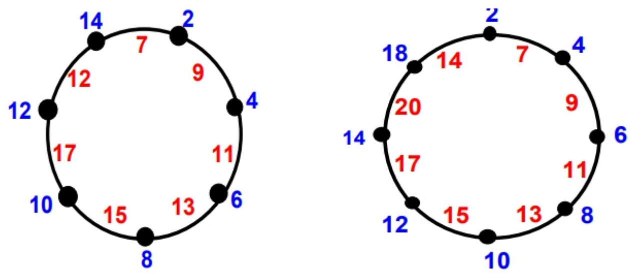

Figure 6. Rough cycle *C7* and *C8*.

4.3 Theorem

The rough star graph $S1,t$ admits even vertex ζ-graceful la beling for $t > 2$.

Proof: Let G be a rough star graph obtained by replacing each vertice of S1,t except the apex vertex u0 .It is the central vertex of the graph G. Let ui be the vertice of rough star graph for $1 \le i \le t$.

The vertex labeling is $f(u0) = 0$ and $f(ui) = 2i$ for $1 \le i \le t$.

The edge labeling is $f(u0ui) = 2i$ for $1 \le i \le t$.

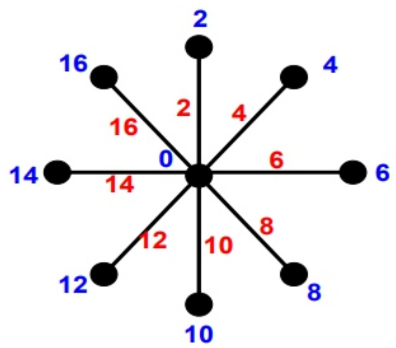

Figure 7. Rough star for *S1,8*.

4.4 Theorem

The rough comb $Pn \odot K1$ admits even vertex ζ-graceful la-belling for all $n \geq 3$.

Proof: *Let v1, v2, . . . , vn* and *u1, u2, . . . , un* be the vertices of $Pn \odot K1$ The general from is given in fig 8. The edge set is defined as follows:

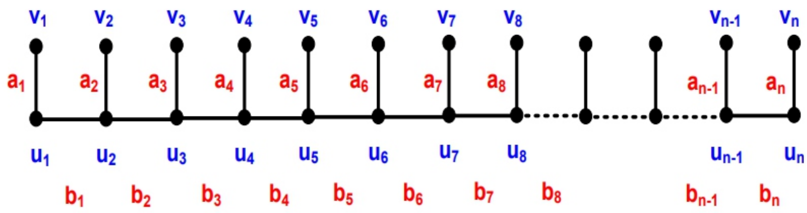

Figure8. Rough comb-$Pn \odot K1$

Case(I): If n is odd, define the vertex labeling as

$f(ui) = 2i$ for $i = 1, 2, . . . n,$ and $f(vi) = 2i + 2n + 2$ *for* $i = 1, 2, . . . n.$

Then the edge labeling is defined as $f(uiui+1) = n + 2i + 1, i = 1, 2, . . . nf(uivi) = 2n + 2i + 1$ *for* $i \in N$. This was shown in Fig. 9.

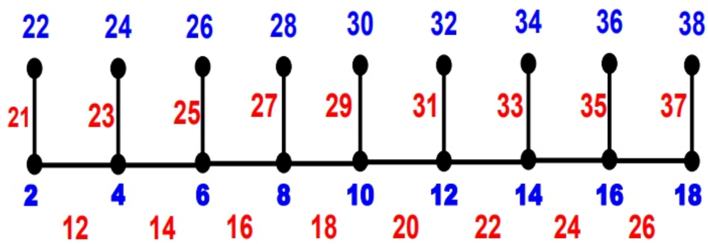

Figure8. Rough comb graph $P9 \odot K1$

Case(ii) : If n is even then the vertex labeling is

$f((ui)) = 2i$ for $1 \leq i \leq n$ and $f(vi) = 2i + 2n$ for $i \in N$.

The edge labeling is defined as follows:

$f\left(u_i u_{i+1}\right)=n+2 i+1$

$f\left(u_i v_i\right)=2 n+2 i$ for $i \in N$. It was represented in Fig. 10.

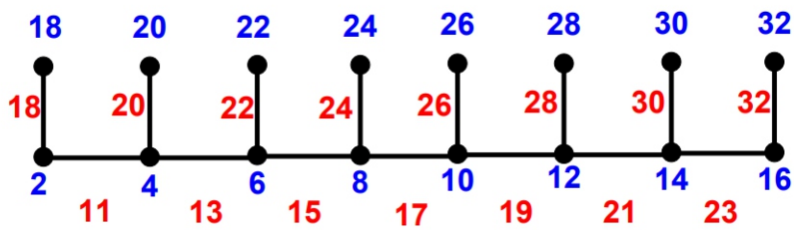

Figure 10. Rough comb graph for $P8 \odot K1$.

4.5 Theorem

The rough ladder graph *Ln* admits even vertex ζ-graceful labeling.

Proof: *Let V (Ln) = {ui, vi/1 ≤ i ≤ n}* be vertex set and *E (Ln) = {uiui+1, vivi+1/1 ≤ i ≤ n − 1}* ∪ *{uivi/1 ≤ i ≤ n}* be the edge set of ladder Ln then it has 2n vertices and *3n − 2* edges as represented in Fig. 11.

Case(I): If n is odd, then the vertex labeling is

f (ui) = 2i for 1 ≤ i ≤ n,

f (vi) = 2i + 2n + 2 for i = 1, 2, . . . n

The edge labeling is defined as

$$f\left(u_i u_{i+1}\right)=n+2i+\left(\frac{n+1}{2}\right), f\left(u_i v_i\right)=2n+2i+\left(\frac{n+1}{2}\right),$$

$$f\left(v_i v_{i+1}\right)=3n+2i+\left(\frac{n+3}{2}\right)+1 \quad \text{for all } i=1,2,\ldots n.$$

Case (ii): If n is even then the vertex labeling is

f(ui) = 2i for *1 ≤ i ≤ n,* and *f(vi) = 2i + 2n* for *i* ∈ *N.*

The edge labeling is

$$f\left(u_i u_{i+1}\right)=n+2i+\left(\frac{n}{2}\right), \ f\left(u_i v_i\right)=2n+2i+\left(\frac{n}{2}\right)-1$$

$$f\left(v_i v_{i+1}\right)=3n+2i+\left(\frac{n}{2}\right) \quad \text{for all } i \in N.$$

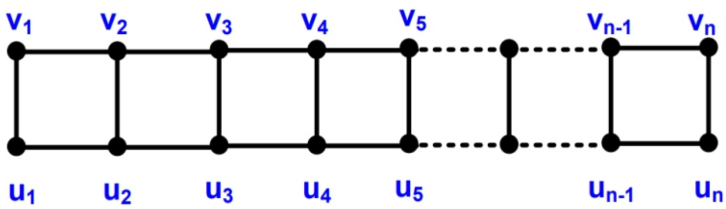

Figure 11. General form of rough ladder graph.

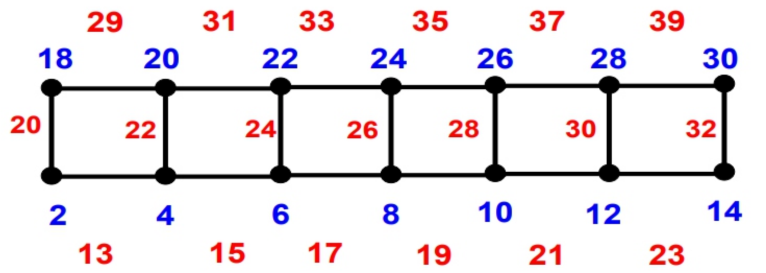

Figure 12. Rough ladder graph *L7*.

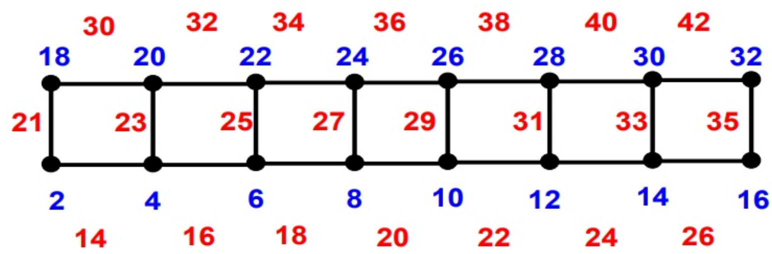

Figure 13. Rough ladder graph L_8.

4.6 Theorem

The rough graph $P_n * S_{1,t}$ is even vertex ζ-graceful labeling.

Proof: Let G be a rough graph obtained by combining path graph and star graph $P_n * S_{1,t}$ or $t = 2$. Let $V(P_n * S_{1,t}) = \{u_i, 1 \leq i \leq n\} \cup \{v_i, a_i, b_i / 1 \leq i \leq n\}$ be vertex set and $E(P_n * S_{1,t}) = \{u_i, u_{i+1} / 1 \leq i \leq n\} \cup \{u_i v_i, v_i a_i, v_i b_i, 1 \leq i \leq n\}$ be the edge set of rough graph for $t = 2$. It was given in Fig. 14.

Case (I): If n is odd, then the vertex labeling is

f(ui) = 4i − 2,f(vi) = 4i,f(ai) = f(vn) − 2 + 4i,f(bi) = f(vn) + 4i.

The edge labeling is defined as follows:

f(uiui+1) = 2n + 4i,f(ui, vi) = 2n + 4i − 1,f(viai) = f(ai) + 1,f(vibi) = f(bi).

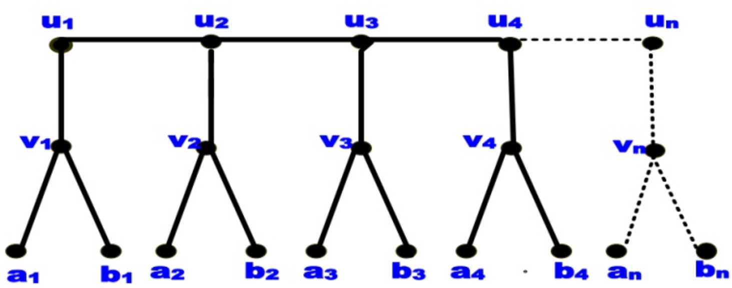

Figure 14. Rough graph $P_n * S_{1,t}$ for $t = 2$.

5 Conclusion

Graph labelling has a wide range of applications in all fields of engineering and science especially in telecommunications.

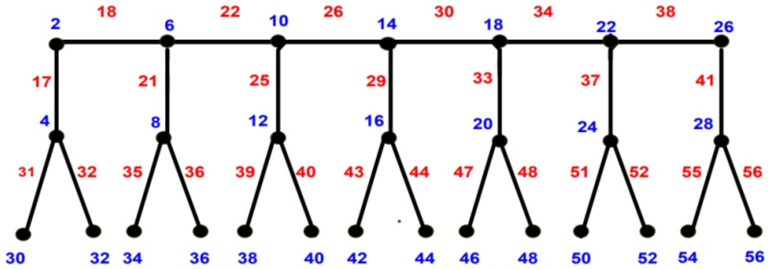

Figure 15. Rough graph *P7*S1,2.*

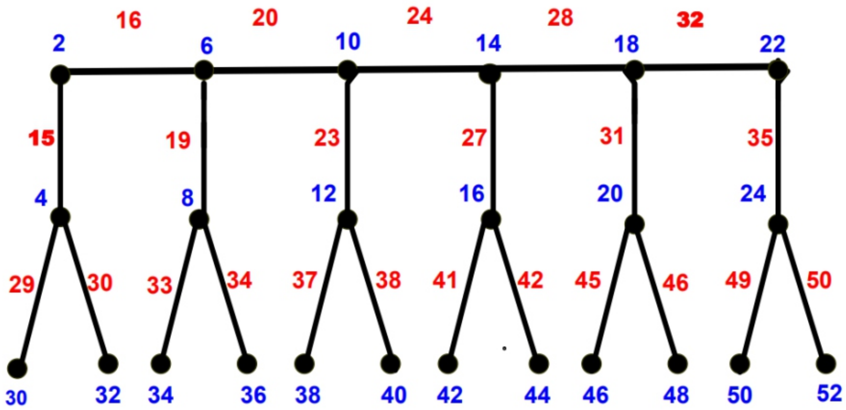

Figure 16. Rough graph $P_6 * S_{1,2}$.

Radio labelling is used to increase the speed of communication in Wireless and sensor networks, Fault tolerant system is designed by using Facility graphs, Voronoi graph is used to cal culate the efficiency of sensor networks. Labelling on Rough graph can be used to identify the strong relationship among ob jects which will enable the decision makers to easily identify the irrelevant conditional features. In this paper, we introduced a new type of labeling called even vertex ζ-graceful labeling on rough graphs. We will extend this labeling for the calculation of consistent features form an Information system by developing new algorithm in future.

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