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Aims and Scope

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Contents

A POISSON ALGEBRA STRUCTURE OVER THE EXTERIOR ALGEBRA OF A QUADRATIC SPACE

Servais Cyr Gatsé1 and Côme Chancel Likouka

A B S T R A C T

We construct a Poisson algebra structure of degree −2 over the exterior algebra of a quadratic space. Here we do not use Clifford algebra as in [4].

1. INTRODUCTION

A graded Lie algebra of degree $-\tau$, where $\tau \geq 0$ is an integer, over a commuta tive field K, is a graded vector space $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} \mathcal{G}^n$ together with a bilinear map

$$[,\,] : G \times G \longrightarrow G,(x, y)\ 7{\longrightarrow}\ [x, y] ,$$

called bracket and which satisfies the following conditions:

(1) $[Gp, Gq] \subset Gp+q-\tau;$

(2) $[x, y] = -(-1)(p-\tau)\cdot(q-\tau)[y, x]$, $x \in Gp, y \in Gq;$

(3) $(-1)(p-\tau)(r-\tau)[x, [y, z]] + (-1)(q-\tau)(p-\tau)[y, [z, x]]+ (-1)(r-\tau)(q-\tau)[z, [x, y]] = 0, x \in Gp, y \in Gq,$ $z \in Gr.$

The identity (3) is equivalent to the following:

$$[x, [y, z]] = [[x, y] , z] + (-1)(p-\tau)(q-\tau)[y, [x, z]] .$$

A commutative algebra structure over G of degree $-\tau$ is the data of a multiplication, denoted by $\cdot$, over G satisfying

$$x \cdot y = (-1)^{(p-\tau)\cdot(q-\tau)} y \cdot x,$$

with $x \in Gp, y \in Gq$.

A Poisson algebra structure of degree $-\tau$ over G is simultaneously the data of a graded Lie algebra structure of degree $-\tau$ and a graded commutative algebra of degree $-\tau$ over G satisfying

$$[x, y \cdot z] = [x, y] \cdot z + (-1)(p-\tau)\cdot qy \cdot [x, z] ,$$

with $x \in Gp, y \in Gq$.

The goal of the present paper is to show that the exterior algebra of a quadratic space admits a Poisson structure of degree −2.

We organize this paper as follows. In Section 2, we present the notion of extension of the Lie bracket. In

Section 3, we recall the definition of a quadratic space. Finally Section 4 deals with Poisson bracket on *(E)*.

2. EXTENSION OF THE LIE BRACKET

Let V be a finite-dimensional (complex or real) vector space, and let V^* be its dual vector space. We consider the exterior algebra of the direct sum of V and V^*

(2.1) $$\bigwedge\left(V\bigoplus V^*\right) = \bigoplus_{n=-2}^{\infty}\Big(\bigoplus_{p+q=n}\big(\bigwedge^{q+1}V^*\bigoplus\bigwedge^{p+1}V\big)\Big).$$

We say that an element of $\bigwedge(V\bigoplus V^*)$ is of bidegree (p,q) and of degree $n=p+q$ if it belongs to $\bigwedge^{q+1}V^*\bigoplus\bigwedge^{p+1}V$. Thus elements of the base field are of bidegree $(-1,-1)$, elements of V (resp. V^*) are of bidegree $(0,-1)$ (resp. $(-1,0)$), and a linear map $\mu:\bigwedge^2 V\longrightarrow V$ (resp. $\gamma: V\longrightarrow\bigwedge^2 V$) can be considered to be an element of $\bigwedge^2 V^*\bigoplus V$ (resp. $V^*\bigoplus\bigwedge^2 V$) which is of bidegree $(0,1)$ (resp. $(1,0)$).

Proposition 2.1. [3] On the graded vector space $\bigwedge(V\oplus V^*)$ there exists a unique graded Lie bracket, called the big bracket, such that

(i) *if $x, y\in V$, $[x, y] = 0$,*
(ii) *if $\zeta, \eta\in V^*$, $[\zeta, \eta] = 0$,*
(iii) *if $x\in V$, $\eta\in V^*$, $[x, \eta] = <\eta, x>$,*
(iv) *if u, v, w* $\bigwedge(V\oplus V^*)$ are of degree $|u|, |v|, |w|$ respectively, then

(2.2) $$[u, v\wedge w] = [u, v]\wedge w + (-1)^{|u||v|}v\wedge[u, w]$$

This last formula is called the graded Leibniz rule. The following proposition lists important properties of the big bracket.

Proposition 2.2. *[3]* Let *$[\cdot,\cdot]$ denote the big bracket. Then*

(i) $\mu:\bigwedge^2 V\longrightarrow V$ *is a Lie bracket if and only if* $[\mu,\mu]=0$.
(ii) ${}^t\gamma:\bigwedge^2 V^*\longrightarrow V^*$ *is a Lie bracket if and only if* $[\gamma,\gamma]=0$.
(iii) *Let $\mathcal{G}=(V,\mu)$ be a Lie algebra. Then γ is a 1-cocycle of $\mathcal{G}$ with values in $\bigwedge^2\mathcal{G}$, where $\mathcal{G}$ acts on $\bigwedge^2\mathcal{G}$ by the adjoint action, if and only if $[\mu,\gamma]=0$.*

By the graded commutativity of the big bracket,

(2.3) $$[\mu, \gamma] = [\gamma, \mu].$$

By the bilinearity and graded skew-symmetry of the big bracket, one has

(2.4) $$[\mu+\gamma, \mu+\gamma] = [\mu, \mu] + 2[\mu, \gamma] + [\gamma, \gamma].$$

Using the bigrading of $\bigwedge(V \oplus V^*)$, we see that the conditions

(2.5) $$[\mu+\gamma, \mu+\gamma] = 0$$

and

(2.6) $$[\mu, \mu] = 0, [\mu, \gamma] = 0, [\gamma, \gamma] = 0$$

are equivalent.

Lemma 2.1. *Let $G = (V, \mu)$ be a Lie algebra. Then:*

(I) The map $d\mu : a \mapsto [\mu, a]$ is a derivation of degree 1 and of square 0 of the graded Lie algebra $\bigwedge(V \oplus V^*)$.

(ii) If $a \in \bigwedge V$, then $d\mu a = -\delta a$, where δ is the Lie algebra cohomology operator.

(iii) For $a, b \in \bigwedge V$, *let us set*

(2.7) $$[[a, b]] = [[a, \mu], b].$$

Then $[[\cdot, \cdot]]$ is a graded Lie bracket of degree *1* on *V* extending the Lie bracket of *G*.

3. QUADRATIC SPACE

In the following E denotes a vector space over a commutative field K with a characteristic different from 2 and $\bigwedge(E) = \bigoplus_{n\in\mathbb{N}} \bigwedge^n(E)$ denotes the exterior algebra of *E*.

Recall that a derivation of $\bigwedge(E)$ of degree r, with $r \in \mathbb{Z}$, is a linear map

$$d : \bigwedge(E) \longrightarrow \bigwedge(E)$$

of degree *r* satisfying

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{p\cdot r}\alpha \wedge d(\beta)$$

for all $\alpha \in \bigwedge^p(E)$ and for all $\beta \in \bigwedge(E)$.

It is the same to say that a linear map

$$d : \bigwedge(E) \longrightarrow \bigwedge(E)$$

is a derivation of degree *r* if and only if *d* is of degree *r* and that

(3.1) $$d(y_1 \wedge \ldots \wedge y_q) = \sum_{j=1}^{q} (-1)^{(j-1)\cdot r}\, y_1 \wedge \ldots \wedge y_{j-1} \wedge d(y_j) \wedge y_{j+1} \wedge \ldots \wedge y_q$$

for all $q \in N$.

Recall that a quadratic form on E is a map $q : E \longrightarrow K$ such that:

1) $q(\lambda \cdot x) = \lambda 2 \cdot q(x), \lambda \in K, x \in E;$

2) the map

$$E \times E \longrightarrow \mathbb{K}, (x, y) \longmapsto \frac{1}{2}\left[q(x+y) - q(x) - q(y)\right],$$

is a symmetric bilinear form.

A quadratic space structure on E is given by a symmetric bilinear form f on E.

In this case we say that the pair (E, f) is a quadratic space.

Proposition 3.1. *If (E, f) is a quadratic space, then the map*

$$qf : E \longrightarrow K, \ x \ 7\longrightarrow f(x, x),$$

is a quadratic form.

Proof. Simple check.

4. POISSON BRACKET ON $\bigwedge$ (E)

In the following (E, f) is a quadratic space. For $x \in E$ and for $q \geq 1$ an integer, we have:

Proposition 4.1. *The map*

(4.1)
$$E^q \longrightarrow \bigwedge^{q-1}(E),$$
$$(y_1, \ldots, y_q) \longmapsto \sum_{j=1}^{q} (-1)^{j-1} f(x, y_j)\, y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q$$

is alternating multilinear. So there is a unique linear map

(4.2)
$$f_x^q : \bigwedge^{q}(E) \longrightarrow \bigwedge^{q-1}(E)$$

such that

(4.3)
$$f_x^q(y_1 \wedge \ldots \wedge y_q) = \sum_{j=1}^{q} (-1)^{j-1} f(x, y_j)\, y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q.$$

Proof. The proof is straightforward.

For $x = 0$, one has $f_x^q = 0$.

We set

$$f_x = f_x^1 + f_x^2 + \cdots + f_x^q + \cdots .$$

Thus $f_x : \bigwedge(E) \longrightarrow \bigwedge(E)$ is a linear map of degree -1 with $f_x \big|_{\bigwedge^q(E)} = f_x^q$.

Proposition 4.2. *The linear map*

(4.4) $$f_x : \bigwedge(E) \longrightarrow \bigwedge(E)$$

is a derivation of degree -1.

Proof. We have

$$\begin{aligned} f_x(y_1 \wedge \ldots \wedge y_q) &= f_x^q(y_1 \wedge \ldots \wedge y_q) \\ &= \sum_{j=1}^{q} (-1)^{j-1} f(x, y_j)\, y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q \\ &= \sum_{j=1}^{q} (-1)^{j-1} y_1 \wedge \ldots \wedge y_{j-1} \wedge f(x, y_j) \wedge y_{j+1} \wedge \ldots \wedge y_q \\ &= \sum_{j=1}^{q} (-1)^{j-1} y_1 \wedge \ldots \wedge y_{j-1} \wedge f_x(y_j) \wedge y_{j+1} \wedge \ldots \wedge y_q . \end{aligned}$$

Considering (3.1), we deduce that f_x is a derivation of degree -1.

For a decomposable element $x_1 \wedge \ldots \wedge x_p \in \bigwedge^p(E)$, $p \geq 1$, we have:

Proposition 4.3. The map

(4.5) $$E^q \longrightarrow \bigwedge^{q-2}(E), (y_1, \ldots, y_q) \longmapsto -(-1)^p \sum_{j=1}^{q} (-1)^{j-1} f_{y_j}^p (x_1 \wedge \ldots \wedge x_p)\, y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q$$

being alternating multilinear, then there exists a unique linear map

(4.6) $$f_{x_1 \wedge \ldots \wedge x_p}^q : \bigwedge^q(E) \longrightarrow \bigwedge^{q-2}(E)$$

such that

(4.7) $$f_{x_1 \wedge \ldots \wedge x_p}^q (y_1 \wedge \ldots \wedge y_q) = -(-1)^p \sum_{j=1}^{q} (-1)^{j-1} f_{y_j}^p (x_1 \wedge \ldots \wedge x_p)\, y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q .$$

Moreover, for $p \geq 1$ and $q \geq 1$, we have

$$f^q_{x_1\wedge\ldots\wedge x_p}(y_1 \wedge \ldots \wedge y_q) = -(-1)^{pq} \cdot f^p_{y_1\wedge\ldots\wedge y_q}(x_1 \wedge \ldots \wedge x_p). \tag{4.8}$$

Proof. The proof of the existence and uniqueness of the linear map $f^q_{x_1\wedge\ldots\wedge x_p}$ is obvious.

On the other hand, for the proof of the last assertion, one has

$$\begin{aligned}
&f^q_{x_1\wedge\ldots\wedge x_p}(y_1 \wedge \ldots \wedge y_q) \\
&= (-1)^p \sum_{i=1}^{p}\sum_{j=1}^{q} (-1)^{i+j-1} f(x_i, y_j) \cdot x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_p \wedge y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q \\
&= (-1)^p \sum_{i=1}^{p}\sum_{j=1}^{q} (-1)^{i+j-1} \cdot (-1)^{(p-1)(q-1)} f(y_j, x_i) \cdot y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q \\
&\quad \wedge x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_p \\
&= -(-1)^{pq} \cdot (-1)^q \sum_{i=1}^{p}\sum_{j=1}^{q} (-1)^{i+j-1} f(y_j, x_i) \cdot y_1 \wedge \ldots \wedge \widehat{y_j} \wedge \ldots \wedge y_q \\
&\quad \wedge x_1 \wedge \ldots \wedge \widehat{x_i} \wedge \ldots \wedge x_p \\
&= -(-1)^{pq} \cdot f^p_{y_1\wedge\ldots\wedge y_q}(x_1 \wedge \ldots \wedge x_p),
\end{aligned}$$

as desired.

We set $f_{x_1\wedge\ldots\wedge x_p} = f^1_{x_1\wedge\ldots\wedge x_p} + f^2_{x_1\wedge\ldots\wedge x_p} + \cdots + f^q_{x_1\wedge\ldots\wedge x_p} + \cdots$. From (4.8), we deduce by linearity the following result:

Corollary 4.1. *For $\alpha \in \bigwedge^p(E)$ and $\beta \in \bigwedge^q(E)$, with $p \geq 1$ and $q \geq 1$, we have:*

$$f_\alpha(\beta) = -(-1)^{p\cdot q} f_\beta(\alpha). \tag{4.9}$$

Thus $f_{x_1\wedge\ldots\wedge x_p} : \bigwedge(E) \longrightarrow \bigwedge(E)$ is a linear map of degree $p-2$ with

$$f_{x_1\wedge\ldots\wedge x_p}\big|_{\bigwedge^q(E)} = f^q_{x_1\wedge\ldots\wedge x_p}.$$

Proposition 4.4. *The linear map*

$$f_{x_1\wedge\ldots\wedge x_p} : \bigwedge(E) \longrightarrow \bigwedge(E)$$

is a derivation of degree $p-2$.

Proof. One has

$$\begin{aligned}
& f_{x_1\wedge\ldots\wedge x_p}\left(y_1\wedge\ldots\wedge y_q\right)\\
= \; & f^q_{x_1\wedge\ldots\wedge x_p}\left(y_1\wedge\ldots\wedge y_q\right)\\
= \; & -(-1)^p\sum_{j=1}^{q}(-1)^{j-1}\, f^p_{y_j}\left(x_1\wedge\ldots\wedge x_p\right)\wedge y_1\wedge\ldots\wedge\widehat{y_j}\wedge\ldots\wedge y_q\\
= \; & -(-1)^p\sum_{j=1}^{q}(-1)^{(j-1)\cdot p}\, y_1\wedge\ldots\wedge y_{j-1}\wedge f^p_{y_j}\left(x_1\wedge\ldots\wedge x_p\right)\wedge y_{j+1}\wedge\ldots\wedge y_q\\
= \; & -(-1)^p\sum_{j=1}^{q}(-1)^{(j-1)\cdot p}\, y_1\wedge\ldots\wedge y_{j-1}\wedge\left[-(-1)^p\, f^1_{x_1\wedge\ldots\wedge x_p}(y_j)\right]\wedge y_{j+1}\\
& \wedge\ldots\wedge y_q\\
= \; & \sum_{j=1}^{q}(-1)^{(j-1)\cdot p}\, y_1\wedge\ldots\wedge y_{j-1}\wedge f_{x_1\wedge\ldots\wedge x_p}(y_j)\wedge y_{j+1}\wedge\ldots\wedge y_q\\
= \; & \sum_{j=1}^{q}(-1)^{(j-1)(p-2)}\, y_1\wedge\ldots\wedge y_{j-1}\wedge f_{x_1\wedge\ldots\wedge x_p}(y_j)\wedge y_{j+1}\wedge\ldots\wedge y_q,
\end{aligned}$$

as required.

We denote, $Der_{\mathbb{K}}\left[\bigwedge(E)\right]$, the space of derivations (of all degrees) of $\bigwedge(E)$.

Proposition 4.5. The map

$$E^p \longrightarrow Der_{\mathbb{K}}\left[\bigwedge(E)\right],\,(x_1,\ldots,x_p)\longmapsto f_{x_1\wedge\ldots\wedge x_p},$$

is alternating multilinear. Thus there exists a unique linear map

(4.10) $$\widetilde{f}^p : \bigwedge^p(E)\longrightarrow Der_{\mathbb{K}}\left[\bigwedge(E)\right]$$

such that

(4.11) $$\widetilde{f}^p\left(x_1\wedge\ldots\wedge x_p\right) = f_{x_1\wedge\ldots\wedge x_p}.$$

Proof. The proof is obvious.

We set $\widetilde{f}=\widetilde{f}^1+\widetilde{f}^2+\cdots+\widetilde{f}^p+\cdots$. So when $\alpha\in\bigwedge^p(E)$, then $\widetilde{f}(\alpha)$ is a derivation of $\bigwedge(E)$ of degree $p-2$.

For α (E) and β (E), we set

(4.12) $\bigwedge$ $\bigwedge$

$$[\alpha, \beta]_f = [\widetilde{f}(\alpha)](\beta).$$

We will, subsequently, show that this bracket defines a Poisson structure of degree−2 on V(E).

Note that when α Vp(E), then

(4.13) $$\widetilde{f}(\alpha) = f_\alpha.$$

By construction, we have:

(4.14) $$\left[\mathbb{K}, \bigwedge(E)\right]_f = 0.$$

Theorem 4.1. *The map*

(4.15) $$\bigwedge(E) \times \bigwedge(E) \longrightarrow \bigwedge(E), (\alpha, \beta) \longmapsto [\alpha, \beta]_f,$$

is bilinear and of degree -2.

Proof. The proof is immediate.

Theorem 4.2. *For* $\alpha \in \bigwedge^p(E)$ *and* $\beta \in \bigwedge^q(E)$*, then*

(4.16) $$[\alpha, \beta]_f = -(-1)^{p \cdot q} [\beta, \alpha]_f.$$

Theorem 4.3. *For* $\alpha \in \bigwedge^p(E), \beta \in \bigwedge^q(E)$ *and* $\gamma \in \bigwedge(E)$*, then*

(4.17) $$[\alpha, \beta \wedge \gamma]_f = [\alpha, \beta]_f \wedge \gamma + (-1)^{p \cdot q} \beta \wedge [\alpha, \gamma]_f.$$

Proof. Since $\widetilde{f}(\alpha)$ is a derivation of degree $p-2$, then we have

$$\begin{aligned} [\alpha, \beta \wedge \gamma]_f &= [\widetilde{f}(\alpha)](\beta \wedge \gamma) \\ &= [\widetilde{f}(\alpha)](\beta) \wedge \gamma + (-1)^{(p-2) \cdot q} \beta \wedge [\widetilde{f}(\alpha)](\gamma) \\ &= [\alpha, \beta]_f \wedge \gamma + (-1)^{p \cdot q} \beta \wedge [\alpha, \gamma]_f. \end{aligned}$$

Hence the result.

Theorem 4.4. *For* $\alpha \in \bigwedge^p(E), \beta \in \bigwedge^q(E)$*, then*

(4.18) $$\left[\widetilde{f}(\alpha), \widetilde{f}(\beta)\right] = \widetilde{f}([\alpha, \beta]_f)$$

where

$$\left[\widetilde{f}(\alpha), \widetilde{f}(\beta)\right] = \widetilde{f}(\alpha) \circ \widetilde{f}(\beta) - (-1)^{p \cdot q} \widetilde{f}(\beta) \circ \widetilde{f}(\alpha).$$

Proof. Taking into account (3.1), for all $z \in E$, we check that

$$\left[\widetilde{f}(\alpha), \widetilde{f}(\beta)\right](z) = \widetilde{f}([\alpha, \beta]_f)(z).$$

The result follows.

Theorem 4.5. *The pair* $(\bigwedge(E), [,]_f)$ *is a Poisson algebra of degree* -2.

Proof. Theorems 4.1, 4.2 and 4.4 mean that the pair $(\bigwedge(E), [,]_f)$ is a graded Lie algebra of degree -2. Theorem 4.3 means that the triple $(\bigwedge(E), [,]_f, \wedge)$ is a Poisson algebra of degree -2. □

As $(\bigwedge(E), [,]_f)$ is a graded Lie algebra, we denote $\widetilde{f}$ by ad_f. Thus we have $[ad_f(\alpha)](\beta) = [\alpha, \beta]_f$ and for $\alpha \in \bigwedge^p(E)$, the linear map

$$ad_f(\alpha) : \bigwedge(E) \longrightarrow \bigwedge(E) \quad (4.19)$$

is simultaneously a derivation (of degree $p-2$) of graded Lie algebra and graded commutative Lie algebra.

An element $M \in \bigwedge^3(E)$ is said to be a proto-Lie bialgebra of the quadratic space (E, f) when $[M, M]_f = 0$. In this case, we say that the quadruple $(\bigwedge(E), [,]_f, \wedge, M)$ is a proto-Lie bialgebra (for further details, we refer to [3] and references therein).

Proposition 4.6. *When the quadruple* $(\bigwedge(E), [,]_f, \wedge, M)$ *is a proto-Lie bialgebra, then the map*

$$ad_f(M) : \bigwedge(E) \longrightarrow \bigwedge(E), P \longmapsto [M, P]_f, \quad (4.20)$$

is a coboundary operator.

Proof. The map $ad_f(M)$ is obviously of degree $+1$. Since $ad_f(M)$ is a derivation of graded Lie algebra, then for $P \in \bigwedge(E)$, we have

$$\begin{aligned} [ad_f(M)]^2(P) &= \Big[M, [M, P]_f\Big]_f \\ &= \Big[[M, M]_f, P\Big]_f + (-1)^{3\times 3}\Big[M, [M, P]_f\Big]_f \\ &= -\Big[M, [M, P]_f\Big]_f \\ &= -[ad_f(M)]^2(P). \end{aligned}$$

We deduce that *[adf (M)]2(P) = 0*. Since P is arbitrary, it follows that *[adf (M)]2 =0*. This means that *adf (M)* is a coboundary operator

For $p \in N$, we denote

$$H_f^p(M) = Ker([ad_f(M)]|_{\bigwedge^p(E)})/Im([ad_f(M)]|_{\bigwedge^{p-1}(E)}) \quad (4.21)$$

the cohomology space of degree p.

Proposition 4.7. *We have:*

(1) $H_f^0(M) = \mathbb{K}$;
(2) $H_f^1(M) = Ker([ad_f(M)]|_{\bigwedge^1(E)})$.

Proof. Simple check. □

When V is a vector space over $\mathbb{K}$ and when V^* is the dual of V, then for $E = V + V^*$, the map

$$E \times E \longrightarrow \mathbb{K}, (v + \phi, w + \psi) \longmapsto \phi(w) + \psi(v), \tag{4.22}$$

is a symmetric bilinear form.

The Poisson bracket over $\bigwedge(E)$ defined by (4.22) is called "Big bracket" [3].

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AN EXAMPLE OF LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS

Servais Cyr Gatsé

A B S T R A C T

Our aim in this paper is to give an example of locally conformally symplectic manifolds.

1. INTRODUCTION

The notion of locally conformally symplectic manifold was introduced in [6] and has been studied extensively by Vaisman and many others (see e.g. [1, 2, 5,10, 13]). Locally conformally symplectic manifolds are generalized phase spaces of hamiltonian dynamical systems since the form of the hamiltonian equations is then preserved by homothetic canonical transformations [13]. We recall that a smooth manifold M is a locally conformally symplectic manifold if there exist a *d*-closed 1-form

$$\alpha : \mathfrak{X}(\mathrm{M}) \longrightarrow C^{\infty}(\mathrm{M}),$$

and a nondegenerate *2*-form

$$\Omega : \mathfrak{X}(\mathrm{M}) \times \mathfrak{X}(\mathrm{M}) \longrightarrow C^{\infty}(\mathrm{M}),$$

such that

$$d\Omega = -\alpha \wedge \Omega,$$

where *d* is the exterior differentiation operator. The 1-form α is called the Lee form [6, 13]. The triple *(M, α, Ω)* is called a locally conformally symplectic manifold. In particular, if α is an exact 1-form on *M,* i.e., α = df for some smooth function f on M then Ω is called globally conformally symplectic form on *M* and it is straightforward to verify that $e - f \cdot \Omega$ is a symplectic form on *M.* The 1-form α is unique. This implies that α is uniquely determined by Ω on a smooth manifold M of dimension at least 4. The dimension of a locally conformally symplectic mani fold has to be even. Since Ω *nis* nowhere vanishing, a locally conformally symplectic manifold possesses a canonic orientation [9]. For first properties and examples of locally conformally symplectic manifolds, we refer the reader to [3, 7, 8, 12]. We organize this paper as follows. In Section 2, we study some properties of the Lichnerowicz-de Rham differential. Section 3 deals with the study of example for locally conformally symplectic manifolds.

2. PROPERTIES OF THE COHOMOLOGY OPERATOR *dα*

A differential form *η* of degree *p* defines a multilinear skew-symmetric function

$$\eta : \underbrace{\mathfrak{X}(\mathrm{M}) \times \cdots \times \mathfrak{X}(\mathrm{M})}_{p\ times} \longrightarrow C^{\infty}(\mathrm{M}).$$

Its exterior derivative $d\eta$ is defined as follows:

$$d\eta : \underbrace{\mathfrak{X}(\mathrm{M}) \times \cdots \times \mathfrak{X}(\mathrm{M})}_{(p+1)\ times} \longrightarrow C^{\infty}(\mathrm{M})$$

is the function defined by the formula

$$\begin{aligned}(d\eta)(X_1, \ldots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i-1} X_i \left[\eta(X_1, \ldots, \widehat{X_i}, \ldots, X_{p+1})\right] \\ &+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1})\end{aligned}$$

for any $X_1, \ldots, X_{p+1} \in \mathfrak{X}(\mathrm{M})$, where the sign $\widehat{}$ indicates the absence of the respective arguments [11].

Proposition 2.1. When *Λ(M)* is the *C∞(M)*-module of differential forms on *M* andwhen d is the exterior differentiation operator then for any $\eta \in \Lambda(M)$, we have

$$d\alpha\eta = d\eta + \alpha \wedge \eta.$$

Corollary 2.1. *The 1-form α is dα-closed if, and only if, α is d-closed.*
Corollary 2.2. *The 1-form α is d-closed if, and only if, $d\alpha \circ d\alpha = 0$.*
Proposition 2.2. *We have the following properties:*

(1) $d\alpha 1 = \alpha$;
(2) $d\alpha(\xi \wedge \gamma) = (d\alpha\xi) \wedge \gamma + (-1)|\xi|\xi \wedge (d\alpha\gamma) - (-1)|\xi\wedge\gamma|\xi \wedge \gamma \wedge d\alpha 1$;

for any ξ and γ homogeneous.
Proof. One uses the Proposition 2.1, we have first

$$d\alpha 1 = d1 + 1 \cdot \alpha = \alpha.$$

And for any ξ and γ homogeneous

$$d\alpha(\xi \wedge \gamma) = (d\xi) \wedge \gamma + (-1)|\xi|\xi \wedge (d\gamma) + \alpha \wedge \xi \wedge \gamma.$$

That ends the proof.
The essential difference between *d* and *dα* is that *dα* does not satisfy a Stokes'theorem. Let us introduce the linear map

$$\tau : C\infty(M) -\!\!\longrightarrow Ham(M), f\ 7\!-\!\!\longrightarrow Xf,$$

where *Ham(M)* is the Lie algebra of hamiltonian vector fields on *M,* for more details see [4].

Theorem 2.1. Define *Iα := {f ∈ C∞(M), dαf = 0}*.

(1) The set Iα is an ideal of the Lie algebra *(C∞(M), {, })* and this ideal is the kernel of the homomorphism τ.

(2) The quotient *C ∞(M)/Iα* is a Lie algebra.

3. STUDY OF THE EXAMPLE OF LOCALLY CONFORMALLY SYMPLECTIC MANIFOLDS

We denote *(e1, e2, ..., e2n)* the canonical basis of *R2n* and *(e*1, e*2, ..., e*2n)* the dualbasis. For *i = 1, 2, ..., 2n, e** Iis the canonical projection

pri: R2n −→ R,(t1, t2, ..., t2n) 7−→ ti.

Let $\alpha_0 = de^*_{2n}$ and $\Omega_0 = \sum_{i=1}^{n} d_{\alpha_0} e^*_i \wedge de^*_{n+i}$.

Proposition 3.1. *For any vector field X on $\mathbb{R}^{2n}$, we have*

$$\begin{aligned} i_X\Omega_0 = &-\sum_{i=1}^{n} X\left(e^*_{n+i}\right) \cdot de^*_i \\ &+\sum_{i=1}^{n}\left(X\left(e^*_i\right) + e^*_i \cdot X\left(e^*_{2n}\right) - \delta_{ni} \cdot \left[\sum_{j=1}^{n} e^*_j \cdot X\left(e^*_{n+j}\right)\right]\right) \cdot de^*_{n+i}. \end{aligned}$$

Proof. Since

$$i_X\Omega_0 = \sum_{i=1}^{n} \Omega_0\left(X, \frac{\partial}{\partial e^*_i}\right) \cdot de^*_i + \sum_{i=1}^{n} \Omega_0\left(X, \frac{\partial}{\partial e^*_{n+i}}\right) \cdot de^*_{n+i},$$

we have

$$\Omega_0\left(X, \frac{\partial}{\partial e^*_i}\right) = \left(\sum_{j=1}^{n} d_{\alpha_0} e^*_j \wedge de^*_{n+j}\right)\left(X, \frac{\partial}{\partial e^*_i}\right) = -X\left(e^*_{n+i}\right)$$

and

$$\begin{aligned} \Omega_0\left(X, \frac{\partial}{\partial e^*_{n+i}}\right) &= \left(\sum_{j=1}^{n} d_{\alpha_0} e^*_j \wedge de^*_{n+j}\right)\left(X, \frac{\partial}{\partial e^*_{n+i}}\right) \\ &= \sum_{j=1}^{n}\left(de^*_j + e^*_j \cdot de^*_{2n}\right)(X) \cdot \delta_{ij} \\ &\quad -\sum_{j=1}^{n}\left(de^*_j + e^*_j \cdot de^*_{2n}\right)\left(\frac{\partial}{\partial e^*_{n+i}}\right) \cdot X\left(e^*_{n+j}\right) \\ &= X\left(e^*_i\right) + e^*_i \cdot X\left(e^*_{2n}\right) - \delta_{ni} \cdot \sum_{j=1}^{n} e^*_j \cdot X\left(e^*_{n+j}\right). \end{aligned}$$

The result follows.

Proposition 3.2. The *2*-form Ω*0* is nondegenerate.

Proof. The map

$$\mathfrak{X}\left(\mathbb{R}^{2n}\right) \longrightarrow \Lambda^1\left(\mathbb{R}^{2n}\right), X \longmapsto i_X \Omega_0$$

is injective. Indeed $i_X\Omega_0 = 0$ implies $X\left(e_{n+i}^*\right) = 0$ for any $i = 1, 2, ..., n$ and $X\left(e_i^*\right) + e_i^* \cdot X\left(e_{2n}^*\right) - \delta_{ni} \cdot \left[\sum_{j=1}^{n} e_j^* \cdot X\left(e_{n+j}^*\right)\right] = 0$ for any $i = 1, 2, ..., n$.

Since $X\left(e_{n+i}^*\right) = 0, i = 1, 2, ..., n$ then $X\left(e_{2n}^*\right) = 0$ and $X\left(e_{n+j}^*\right) = 0$ for all $j = 1, 2, ..., n$. We deduce that $X\left(e_i^*\right) = 0$ for $i = 1, 2, ..., n$, so $X = 0$.

The map

$$\mathfrak{X}\left(\mathbb{R}^{2n}\right) \longrightarrow \Lambda^1\left(\mathbb{R}^{2n}\right), X \longmapsto i_X \Omega_0$$

is surjective.

For $\vartheta \in \Lambda^1\left(\mathbb{R}^{2n}\right)$, we verify that if

$$Y = \sum_{i=1}^{n}\left[\vartheta\left(e_{n+i}^*\right) + e_i^* \cdot \vartheta\left(e_n^*\right) - \delta_{ni} \cdot \left(\sum_{j=1}^{n} e_j^* \cdot \vartheta\left(e_j^*\right)\right)\right] \cdot \frac{\partial}{\partial e_i^*} - \sum_{i=1}^{n} \vartheta\left(e_i^*\right) \cdot \frac{\partial}{\partial e_{n+i}^*}$$

we obtain

$$i_Y \Omega_0 = \vartheta.$$

The proof is complete. □

Proposition 3.3. *We get*

$$d_{\alpha_0}\left(\Omega_0\right) = 0.$$

Proof. Since

$$\begin{aligned} d_{\alpha_0}\left(\Omega_0\right) &= d_{\alpha_0}\left(\sum_{i=1}^{n} d_{\alpha_0} e_i^* \wedge d e_{n+i}^*\right) \\ &= -\sum_{i=1}^{n}\left[d_{\alpha_0} e_i^* \wedge d_{\alpha_0}\left(d e_{n+i}^*\right) + \alpha_0 \wedge d_{\alpha_0} e_i^* \wedge d e_{n+i}^*\right] \\ &= 0, \end{aligned}$$

as desired. [

Theorem 3.1. *The triple* $\left(\mathbb{R}^{2n}, \alpha_0, \Omega_0\right)$ *is a locally conformally symplectic manifold.*

Proof. Indeed

$$d\alpha_0 = d\left(d e_{2n}^*\right) = d^2\left(e_{2n}^*\right) = 0.$$

This completes the proof.

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A NOTE ON REFLEXIVE RINGS

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A B S T R A C T

Mason introduced the reflexive property for ideals and then this con cept was generalized by Kim and Baik, defining idempotent reflexive right ideals and rings. In this note we consider reflexive property of a special subring of the infinite upper triangular matrix ring over a ring R. We proved that, if R is a left AP P-ring, then Vn® is reflexive. We also give an example which shows that the ring Vn® need not be left AP P when R is a left AP P-ring.

All rings considered here are associative with identity. Mason introduced the reflexive property for ideals, and this concept was generalized by some authors, defining idempotent reflexive right ideals and rings, completely reflexive rings, weakly reflexive rings (see namely, [1–4]). The reflexive right ideal concept is also specialized to the zero ideal of a ring, namely, a ring *R* is called reflex ive [2] if its zero ideal is reflexive and a ring *R* is called completely reflexive if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Completely reflexive rings are called re versible by Cohn in [5] and also studied in [6]. It is clear that every reduced ring (*i.e*. rings without nonzero nilpotent elements) are completely reflexive and every completely reflexive ring is semicommutative. The notion of Armendariz ring is introduced by Rege and Chhawchharia (see [7]). They defined a ring *R* to be Armendariz *if* $f(x)g(x) = 0$ implies aibj = 0, for all polynomials $f(x) =a0 + a1x + a2x2 + \cdots + amxm$, $g(x) = b0 + b1x + b2x2 + \cdots + bnxn \in R[x]$.

In [8] A ring R is called strongly reflexive whenever $f(x), g(x) \in R[x]$ satisfy $f(x)R[x]g(x) = 0,$ then $g(x)R[x]f(x) = 0.$ Clearly, every strongly reflexive ring is reflexive, but the converse is not true (see [8, Example 2.1]). Obviously, sub rings and direct products of a strongly reflexive ring are strongly reflexive. The concept of quasi-Armendariz rings is another generalization of Armendariz rings. According to [9], a ring R is called a quasi-Armendariz if whenever polynomials $f(x) = a0 + a1x + a2x2 + \cdots + amxm$, $g(x) = b0 + b1x + b2x2 + \cdots + bnxn \in R[x]$ *sat isfy* $f(x)R[x]g(x) = 0,$ then $aiRbj = 0$ for each i, j. It was proved in [6, Proposition 2.4] that if *R* is an Armendariz ring, then *R* is completely reflexive if and only if *R[x]* is completely reflexive. According to [8], if R is quasi-Armendariz, then R is a reflexive ring if and only if *R[x]* is strongly reflexive ring.

Let R be a ring. It was shown in [4] that R is a reflexive ring if and only if *Mn*® is a reflexive for all $n \geq 1$. Here we consider the following ring:

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, a_3, \ldots, a_n \in R \right\}.$$

The aim of this note, we will show that if *R* is a left *AP P*-ring, then *Vn®* is reflexive. We also give an example which shows that the ring *Vn®* need not be left *AP P* when *R* is a left *AP P*-ring.

An ideal *I* of *R* is said to be right s-unital if, for each *a* ∈ *I* there exists an element x ∈ I such that *ax = a.* It follows from Tominaga ([10, Theorem 1]) that I is right s-unital if and only if for any finitely many elements *a1, a2, . . . , an* ∈ *I,* there exists an element *x* ∈ *I* such that *ai = xai(resp. ai = aix)* for each *i = 1, 2, . . . , n.* According to [11] a ring R is called a left AP P-ring if the left annihilator *lR(Ra)* is right s-unital as an ideal of R for any element a ∈ R. Right AP P-rings can be defined analogously. Recall a ring R is a left p.q.-Baer ring if the left annihilator of a principal left ideal of *R* is generated by an idempotent (see, for example, [12–14]). Clearly every left *p.q.*-Baer ring is a left *AP P*-ring (thus the class of left *AP P*-rings includes all biregular rings and all quasi-Baer rings). *A* ring R is a right *P P*-ring if the right annihilator of an element of R is generated by an idempotent. Right P P rings are left *AP P*.

The following results follows from [9,15], respectively.

Proposition 1. *Every left AP P-ring is quasi-Armendariz, but not conversely.*

Lemma 1. *Let R be a left AP P-ring and a1, . . . , an, b1, . . . , bm belong to R. If aiRbj =0 for all i and j, then there exists e* ∈ *R such that ai = aie and eRbj = 0 for all I and j.*

Theorem 1. *Let R be a reduced ring. If R is a left AP P-ring, then Vn® is reflexive.*

Proof. Suppose that R is left APP and $\sum_{i=1}^{\ell} A_i x^i, \sum_{j=1}^{m} B_j x^j \in V_n(R)[x]$ such that $(\sum_{i=1}^{\ell} A_i x^i) V_n(R)[x] (\sum_{j=1}^{m} B_j x^j) = 0$. We will show that

$$(\sum_{j=1}^{m} B_j x^j) V_n(R)[x] (\sum_{i=1}^{\ell} A_i x^i) = 0$$

for all i and j. Suppose that

$$A_i = \begin{pmatrix} a_1^i & a_2^i & a_3^i & a_4^i & \cdots \\ 0 & a_1^i & a_2^i & a_3^i & \cdots \\ 0 & 0 & a_1^i & a_2^i & \cdots \\ 0 & 0 & 0 & a_1^i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B_j = \begin{pmatrix} b_1^j & b_2^j & b_3^j & b_4^j & \cdots \\ 0 & b_1^j & b_2^j & b_3^j & \cdots \\ 0 & 0 & b_1^j & b_2^j & \cdots \\ 0 & 0 & 0 & b_1^j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Set $f_p = \sum_{i=1}^{\ell} a_p^i x^i, g_p = \sum_{j=1}^{m} b_p^j x^j$ for any p with $1 \leq p$. Then from $(\sum_{i=1}^{\ell} A_i x^i)$ $V_n(R)[x](\sum_{j=1}^{m} B_j x^j) = 0$ it follows that for any $\lambda_p = \sum_{k=1}^{h} c_p^k x^k \in R[x]$ with $1 \leq p$.

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 & \cdots \\ 0 & f_1 & f_2 & f_3 & \cdots \\ 0 & 0 & f_1 & f_2 & \cdots \\ 0 & 0 & 0 & f_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots \\ 0 & \lambda_1 & \lambda_2 & \lambda_3 & \cdots \\ 0 & 0 & \lambda_1 & \lambda_2 & \cdots \\ 0 & 0 & 0 & \lambda_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g_1 & g_2 & g_3 & g_4 & \cdots \\ 0 & g_1 & g_2 & g_3 & \cdots \\ 0 & 0 & g_1 & g_2 & \cdots \\ 0 & 0 & 0 & g_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

Note that $a_i c_k b_j x^{i+k+j} = 0$ for all i, j and k with $i + k + j = n$. Since $f \lambda g = 0$, we have the following equations:

$$a_1 c_1 b_1 = 0 \tag{1}$$

$$a_1 c_1 b_2 + a_1 c_2 b_1 + a_2 c_1 b_1 = 0 \tag{2}$$

$$a_1 c_1 b_3 + a_1 c_2 b_2 + a_1 c_3 b_1 + a_2 c_1 b_2 + a_2 c_2 b_1 + a_3 c_1 b_1 = 0 \tag{3}$$

$$\vdots$$

$$a_1 c_1 b_m + a_1 c_2 b_m + \cdots + a_1 c_{m+1} b_1 + \cdots + a_m c_1 b_2 + a_m c_2 b_1 + a_{m+1} c_1 b_1 = 0 \tag{4}$$

$$\vdots$$

$$a_1 c_1 b_{n-1} + a_1 c_2 b_{n-2} + \cdots + a_{n-2} c_2 b_1 + a_{n-1} c_1 b_1 = 0 \tag{5}$$

$$a_1 c_1 b_n + a_1 c_2 b_{n-1} + \cdots + a_{n-1} c_1 b_2 + a_{n-1} c_2 b_1 + a_n c_1 b_1 = 0, \tag{6}$$

where $1 \leq m \leq n$. Note that R is reflexive and that $aRcRc = 0$ if and only if $aRc = 0$ for a, $c \in R$. We freely use these facts in the following computations. From Eq. (1), we have $a1Rb1 = 0$. Thus by Lemma 1, there exist $e \in R$ such that a I = aIeand eRbj = 0 for all i, j and so $f = fe$ *and* $eR[x]g = 0$. Hence $gj \in rR(dR[x])$ for j

$= 2$, *where* $d \in R$ is an arbitrary element. By hypothesis, $rR(dR[x])$ is s-unital and hence by Lemma 1, again there exist $e \in rR(dR[x])$ such that $gj = egj$, *for* $j = 2$. Since dRe = 0, $f1R[x]eg1 = 0$. *Thus* $f1R[x]g1 =$ 0. Multiplying Eq. (2) by Rb1 on the right side, we get a2Rb1Rb1 = 0 and so a2Rb1 = 0. Then Eq. (2) implies a1c1b2 = 0. Substitute et for c1 in a1c1b2 = 0 to yield a1$(et)b2 = 0$, $t \in R$ is an arbitrary element, then we have a1Rb2 = 0. Thus by Lemma 1 again, there exist $u \in R$ such that $a I = a iu$ and $uRbj = 0$ for all i and j. Hence $f = fu$ and $uR[x]g = 0$, $uR[x]g2 = 0$. *Thus* $f1R[x]g2 = 0$ *and so* $f2R[x]g1 = 0$.

Now Eq. (3) becomes

$$a1c1b3 + a2c1b2 + a3c1b1 = 0.$$

Multiply this equality on the right side by $Rb1$ and $Rb2$ in turn, to obtain $a3Rb1 = 0$, $a2Rb2 = 0$ *and* $a1Rb3 = 0$. Thus by Lemma 1, there exist h ∈ R such that $a I = a ih$ *and* $hRbj = 0$ and so f = fh, hR[x]g = 0. Thus $f3R[x]g1 = 0$. By Lemma 1 again, there exist w ∈ R such that a $I = a iw$, $wRbj = 0$, bj ∈ rR(wR) is s-unital and so $f = fw$, $wR[x]g = 0$. Thus $f2R[x]g2 = 0$ *and* $f1R[x]g3 = 0$. Summarizing, we have

that

$$aiRbj = 0 \text{ for } i + j = 2, 3, 4.$$

Inductively, we assume that $aiRbj = 0$ *for* $i + j = 2, 3, \ldots m$ with $m - 1 \leq n$. Then Eq. (4) becomes

$$a1c1bm-1 + a2c1bm + a2c1bm-1 + \cdots + amc1b2 + am-1c1b1 = 0 \ (7).$$

Multiplying Eq. (7) on the right side by $Rb1, Rb2, \ldots$, and Rbm in turn, we obtain $am-1Rb1 = 0$, $amRb2 = 0, \ldots$, and $a2Rbm = 0$, entailing $a1Rbm-1 = 0$. These show that $aiRbj = 0$ for all i and j with $i + j = m - 1$. Consequently, $aiRbj = 0$ for all i and j with $1 \leq i + k \leq n$. Since R is reflexive, $bjRai = 0$ for all i and k with 1 ≤ i + k ≤ n. Hence there exists r ∈ R be an arbitrary element such that $a I = a I r$ and $rRbj = 0$ for all i and j. Hence b j ∈ rR(rR). By hypothesis, $rR(rR)$ is left sunital and by Lemma 1, again which implies that fp = fpr and $rR[x]gp = 0$. Hence $gp \in rR(rR[x])$ for p = 1, 2, . . . is left s-unital. Thus by the induction hypothesis, $g1R[x]f1 = 0$, $g1R[x]f2 = 0$, $g2R[x]f1 = 0, \ldots, g1R[x]fn = 0, \ldots, gnR[x]f1 = 0$. This yields $g\lambda f = 0$, proving that *Vn®* is reflexive.

Proposition 2. *If Vn® is reflexive then so is R.*

Proof. Suppose that $f = \Sigma aixI$, $g = \Sigma bjxj$ *are in* $R[x]$ *such that* $fR[x]g = 0$. *Thenfor any* $\lambda \in R[x]$,

$$\begin{pmatrix} f & 0 & 0 & 0 & \cdots \\ 0 & f & 0 & 0 & \cdots \\ 0 & 0 & f & 0 & \cdots \\ 0 & 0 & 0 & f & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 & 0 & \cdots \\ 0 & \lambda & 0 & 0 & \cdots \\ 0 & 0 & \lambda & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g & 0 & 0 & 0 & \cdots \\ 0 & g & 0 & 0 & \cdots \\ 0 & 0 & g & 0 & \cdots \\ 0 & 0 & 0 & g & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

Thus

$$\begin{pmatrix} b_j & 0 & 0 & 0 & \cdots \\ 0 & b_j & 0 & 0 & \cdots \\ 0 & 0 & b_j & 0 & \cdots \\ 0 & 0 & 0 & b_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} (V_n(R)) \begin{pmatrix} a_i & 0 & 0 & 0 & \cdots \\ 0 & a_i & 0 & 0 & \cdots \\ 0 & 0 & a_i & 0 & \cdots \\ 0 & 0 & 0 & a_i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = 0.$$

for all i and j, which implies that $b_j R a_i = 0$ for all i, j.

Corollary 1. *Let R be a ring. If R is quasi-Armendraiz, then Vn® is reflexive.*

The following example shows that the left *AP P* property of *R* does not imply the left AP P property of *Vn®.*

Example 1. Let *F* be a field and consider the ring *Vn(F)*. Let

$$B = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

belong to $V_n(F)$. *Then* $V_n(F)B = \left\{ \begin{pmatrix} 0 & b & b & b & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mid b \in F \right\}$. *Thus it is easy to see that*

$$l_{V_n(F)}(V_n(F)B) = \left\{ \begin{pmatrix} 0 & x_2 & x_3 & x_4 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mid x_i \in F \right\}.$$

Now let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in l_{V_n(F)}(V_n(F)B).$$

If $V_n(F)$ is left APP, then there exists $C \in l_{V_n(F)}(V_n(F)B)$ such that $A = AC$. But this contradicts with the fact

$$AC = A \begin{pmatrix} 0 & c_2 & c_3 & \cdots \\ 0 & 0 & c_3 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_3 & c_4 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus Vn(F) is not left AP P.

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THE HIGHER FINITE DIFFERENCE METHOD FOR SOLVING THE DYNAMICAL MODEL OF COVID-19

Amar Megrous

ABSTRACT

In the present paper, the SIR model tracks the numbers of susceptible, infected and recovered individuals during an epidemic with the help of ordinary differential equations (ODE). First, we give the model formulation of our phenom ena. Secondly, a fully discrete difference scheme is derived for the SIR model.At the end of this aper, we give the simulation results of the model. A comparison of the obtained numerical results of both the models is performed in the absence of an exact solution.

1. INTRODUCTION

The novel human coronavirus disease 2019 (COVID-19) was first reported in Wuhan, China, in 2019, and subsequently spread globally to become the fifth doc umented pandemic since the 1918 flu pandemic. By September 2021, almost two years after COVID-19 [1] and [2] was first identified, there had been more than 200 million confirmed cases and over 4.6 million lives lost to the disease. Here, we take an in-depth look at the history of COVID-19 from the first recorded case to the current efforts to curb the spread of the disease with worldwide vaccination programs.

The first official cases of COVID-19 were recorded on the 31st of December,2019, when the World Health Organization (WHO) was informed of cases of pneu monia in Wuhan, China, with no known cause. On the 7th of January, the Chinese authorities identified a novel coronavirus, temporally named 2019-nCoV, as the cause of these cases. Weeks later, the WHO declared the rapidly spreading COVID-19 outbreak as a Public Health Emergency of International Concern on the 30th of January 2020. It wasn't until the following month, however, on the 11th of February that the novel coronavirus got its official name - COVID-19. Nine days later, the US Centers for Disease Control and Prevention (CDC) confirmed the first person to die of COVID-19 in the country. The individual was a man in his fifties who lived in Washington state.

A finite difference method [6]- [12] proceeds by replacing the derivatives in the differential equations by finite difference approximations. This gives a large algebraic system of equations to be solved in place of the differential equation [14]- [18], something that is easily solved on a computer.

Mathematical modeling can be thought of as an iterative process made up of the following components. (Note that the word tep is intentionally avoided to highlight the lack of a prescribed ordering of these components, as some may occur simultaneously and some may be repeated.)

The remainder of this paper is structured as follows. Section 2 discusses the formulation of the model. In the section 3 we present the forward second order accurate approximation to the first derivative. In the section 3 we present the forward second order accurate approximation to the first derivative. In section 4 we propose a new numerical scheme for a spatially discrete model of total variation of indice *i*. Finally, in the last section, We give some numerical results including both simulation and an empirical example to study the proposed testing procedure in different times.

2. MODEL FORMULATION

The COVID-19 pandemic, among other pandemics from the past, has attracted great attention not only from mathematicians but researchers from numerous fields. It is assumed that the sum of the four categories S,I,R is equal to the total population (M) at time t=0 (system parameters relate to the time t in days). Be sides, nowadays the researchers are devoting their research work to the fractional order COVID-19 mathematical models. A huge number of good research papers related to fractional-order COVID-19 mathematical models can be found in the literature, some of them are the following [1]- [2].
For nonlinear systems, we consider the effects of three unknown functions on each other. A three by three system of nonlinear ordinary differential equations has the form:

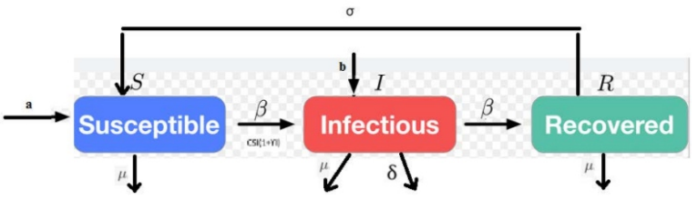

FIGURE 1. The Model of SIR

This is because of two exposures over a small time period: a single contact produces infection at the rate *CIS*, while the new infective individuals arise from double exposures with *CI2S*. It produces further chance that the recovered individual againmay catch infection.
Here we remark that the function *Φ(S, I) = CI(t)S(t)(1 + γI(t)),* where both *C, γ* are positive constants. This is an interesting example for nonlinear incidence rate already used by some authors [17, 31, 32].
The dynamics of the population are described by the following differential equations:

$$\begin{aligned} \frac{dS(t)}{dt} &= a - CI(t)\big(1+\gamma I(t)\big) - \mu S(t) + \alpha R(t), \\ \frac{dI(t)}{dt} &= CI(t)S(t)\big(1+\gamma I(t)\big) - (\beta+\mu+\delta-b)I(t), \\ \frac{dR(t)}{dt} &= \beta I(t) - (\alpha+\mu)R(t). \end{aligned} \tag{2.1}$$

The parameters involved in model (1) are described as in Table 1.

3. FORWARD SECOND ORDER ACCURATE APPROXIMATION TO THE FIRST DERIVATIVE

Develop a forward difference formula for *f(1)I* which is *E = O(h)2* accurate. First derivative with *O(h)* accuracy then the minimum number of nodes is 2. Then, the first derivative with *O(h)* accuracy then need 3 nodes

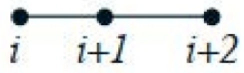

FIGURE 2. 3 NODES

The first forward derivative can therefore be approximated to O(h) as:

$$\left.\frac{df}{dx}\right|_{x=x_i} - E = \frac{\alpha_1 + \alpha_2 f_{i+1} + \alpha_3 f_{i+2}}{h}.$$

The T.S. expansions about x_i are:

$$f_i = f_i,$$
$$f_{i+1} = f_i + hf_i^{(1)} + \frac{h^2}{2}f_i^{(2)} + \frac{h^3}{6}f^{(3)} + O(h)^4,$$
$$f_{i+2} = f_i + 2hf_i^{(1)} + 2h^2 f_i^{(2)} + \frac{4}{3}h^3 f_i^{(3)} + O(h)^4.$$

We substituting into our assumed form of and re-arranging

$$\frac{\alpha_1 + \alpha_2 f_{i+1} + \alpha_3 f_{i+2}}{h} = \frac{(\alpha_1 + \alpha_2 + \alpha_3)}{h} f_i + (\alpha_2 + 2\alpha_3) f_i^{(1)} + \left(\frac{\alpha_2}{2} + 2\alpha_3\right) h f_i^{(2)} + \left(\frac{1}{6}\alpha_2 + \frac{4}{3}\alpha_3\right) h^2 f_i^{(3)} + O(h)^3.$$

Desire $f_i^{(1)}$ and 2^{nd} order accuracy then coefficient of $f_i^{(1)}$ must equal unity and coefficients of f_i and $f_i^{(2)}$ must vanish

$$\frac{\alpha_1 + \alpha_2 + \alpha_3}{h} = 0,$$
$$(\alpha_2 + 2\alpha_3) = 1,$$
$$\left(\frac{\alpha_2}{2} + 2\alpha_3\right) h = 0.$$

We solve these simultaneous equations

$$\alpha_1 = -\frac{3}{2}, \quad \alpha_2 = 2, \quad \alpha_3 = -\frac{1}{2}.$$

Thus the equation now becomes

$$\frac{-\frac{3}{2}f_i + 2f_{i+1} - \frac{1}{2}f_{i+2}}{h} = (0)f_i + (2-1)f_i^{(1)} + (0)f_i^{(2)} + \left(\frac{1}{6}.2 - \frac{4}{3}.\frac{1}{2}\right) h^2 f_i^{(3)} + O(h)^3,$$

then we get

$$f_i^{(1)} = \frac{-3f_i + 4f_{i+1} - f_{i+2}}{2h} + \frac{1}{3}h^2 f(3) + O(h)^3.$$

The forward difference approximation of 2nd order accuracy

(3.1) $$f_i^{(1)} = \frac{-3f_i + 4f_{i+1} - f_{i+2}}{2h} + E \text{ where } E = \frac{1}{3}h^2 f_i^{(3)}.$$

4. THE DISCRETE MODEL

A finite difference method proceeds by replacing the derivatives in the differen tial equations by finite difference approximations. This gives a discrete model as fellows:

(4.1)
$$\begin{aligned}
\frac{-3S_i + 4S_{i+1} - S_{i+2}}{2h} &= a - CI_i\big(1 + \gamma I_i\big) - \mu S_i + \alpha R_i,\\
\frac{-3I_i + 4I_{i+1} - I_{i+2}}{2h} &= CI_iS_i\big(1 + \gamma I_i\big) - (\beta + \mu + \delta - b)I_i,\\
\frac{-3R_i + 4R_{i+1} - R_{i+2}}{2h} &= \beta I_i - (\alpha + \mu)R_i.
\end{aligned}$$

After arrangement of the previous equations, we obtain:

(4.2)
$$\begin{aligned}
S_{i+2} &= -3S_i + 4S_{i+1} - 2h(a - CI_i\big(1 + \gamma I_i\big) - \mu S_i + \alpha R_i,\\
I_{i+2} &= -2h(CI_iS_i\big(1 + \gamma I_i\big) - (\beta + \mu + \delta - b)I_i) - 3I_i + 4I_{i+1},\\
R_{i+2} &= -2h(\beta I_i - (\alpha + \mu)R_i) - 3R_i + 4R_{i+1}.
\end{aligned}$$

The initial conditions (ICs) for the above model are given as follows: $S(0) \geq 0, I(0) \geq 0$ and $R(0) \geq 0$.

5. NUMERICAL RESULTS

In this section, we present some numerical results obtained by applying the new methods. These results indicate the efficiency of the methods. Consider model (4.2) with the parameters given in Figure 3.

Using the differential equations of the SIR model and converting them to numerical discrete forms, one can set up the recursive equations and calculate the S, I, and R populations with any given initial conditions but accumulate errors over a long calculation time from the reference point. Sometimes a convergence test is needed to estimate the errors. Given a set of initial conditions and the disease spreading data, one can also fit the data with the SIR model and pull out the three reproduction numbers when the errors are usually negligible due to the short times

Parameters	The physical interpretation
$S(t)$	Susceptible compartment
$I(t)$	Infected compartment
$R(t)$	Recovered compartment
a	The recruitment rate
μ	Natural death
δ	Death due to corona
b	The immigration rate of infected individuals
β	Corona infection recovery rate
C	The infection rate
γ	Rate at which recovered individuals lose immunity
α	Rate of recovery from infection

FIGURE 3. The Parameters in our Model

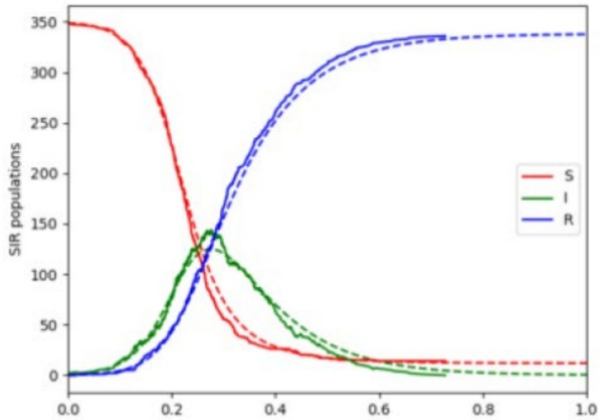

FIGURE 4. Time arbitrary units

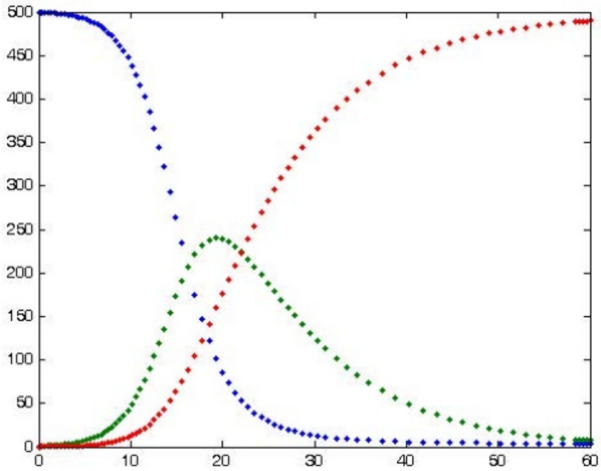

FIGURE 5. Time days between 0 to 60

step from the reference point. Let us now implement the model in MATLAB, using the ode45 command to numerically solve differential equations. The script SIR.m provided on the web page will also help you to plot the results as in Fig. 4 and Fig. 5 with runing the model with the preset parameters.

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