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COMPLEMENTARY TRIPLE CONNECTED AT MOST TWIN DOMINATION NUMBER OF A GRAPH

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ABSTRACT

In this article, we introduce the concept of complementary triple connected at most twin domination number of a graph. A set S V is called a complementary triple connected at most twin dominating set (CTATD(G)),if every vertex $v \in V - S; 1 \leq |N(v) \cap S| \leq 2$ and $\langle V - S \rangle$ *is triple connected. The minimum cardinality taken over all the complementary triple connected at most twin dominating sets in G is called the complementary triple connected at most twin domination number of G and is denoted by CTATD(G). In this article we investigate this parameter for some standard and special types of graphs.*

Keywords: triple connected, [1, 2] dominating set, triple connected domination number.

1. Motivation

Mustapha Chellali et al. [4] first studied the concept of [1, 2] set. Xiaojing Yang and Baoyin-dureng Wu [2] extended the study of this parameter. G. Mahadevan et al. developed the theory of 1,2 cc [5] and the concept of at most twin outer perfect domination number of a graph [3]. Paulraj Joseph et al., [6] were introduced the triple connected graphs. Keeping all the above definitions as the motivation we keep the dominating set to be [1, 2] - dominating set and its complement to be triple connected, thereby we introduce a new domination parameter called CTATD-number of a graph.

2. Preliminaries

For our further discussion, we mention the following definitions which are available in [1]. A Helm graph *Hp* is a graph obtained from the wheel *W1,n* by joining a pendent vertex to each vertex in the outer cycle of *W1, n* Subdivide every edge in the graph *G*, join the vertices that are adjacent in *G,* and join the subdivided vertices that are adjacent to a common vertex. The obtained graph is called total graph. The flower graph *Flp* is the graph obtained from the Helm *Hp* by joining each pendant vertex to the apex of the helm. A closed Helm Chnis the graph obtained from a Helm *Hn* by joining each pendant vertex to form a cycle. The barbell graph *K K e p p* +is obtained by joining two copies of *Kp* by a bridge. The friendship graph, denoted by *Fp* can be constructed by identifying p copies of the cycle *C3* at a common vertex. Subdivide every edge in the graph *G*, join subdivided vertices that are adjacent to a common vertex, the obtained graph is called middle graph *M(G)*.The triangular snake graph *Tsp* is obtained from a path (α1, α2.....αp) by joining aiand *ai+1* to a new vertex bifor *i = 1, 2, , n − 1*.That is every edge of a path is replaced by a triangle . *C3* The mirror graph is . *Mr= P2 G* Central graph is obtained by subdividing every edge and obtain the original graph *G.* The shadow graph D (G) 2of a connected graph G is obtained by taking two copies of *G* say *G* and *G* join each vertex *u* in *G* to the neighbours of corresponding vertex *u*" in *G*"

3. Complementary Triple Connected at Most Twin Domination Number of a Graph

Definition 3.1. A set *S U V* is called a complementary triple connected at most twin dominating set *(CTATD(G))* if every vertex $v \in V - S; 1 \leq N(v) \; S \leq 2$ and *V − S* is triple connected. The minimum cardinality of a *CTATD*-set is called the complementary triple connected at most twin domination number (*CTATD*-number) and is denoted by *CTATD(G)*.

Observation 3.1. Complementary triple connected at most twin domination number does not exist for path, cycle, star graph, bistar graph, friendship graph.

Observation 3.2. For any connected graph G, $\gamma(G) \leq \gamma_{[1,2]cc}(G) \leq CTATD(G)$ and the bounds are sharp.

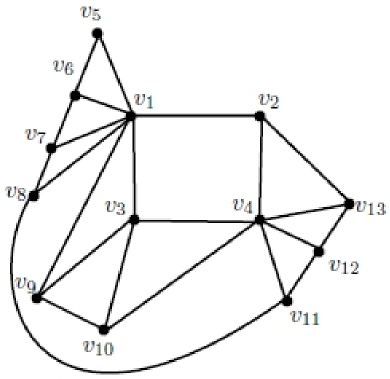

Figure 3.3.

In figure 3.3, $S_1 = \{v_1, v_4\}$ is a dominating set of smallest size, so that $\gamma(G) = 2$.

$S_2 = \{v_1, v_3, v_4, v_9, v_{10}\}$ is a $[1, 2]cc$ dominating set of minimum cardinality, so that $\gamma_{[1,2]cc}(G) = 4$.

$S_3 = \{v_1, v_3, v_4, v_9, v_{10}\}$ is a complementary triple connected at most twin dominating set of minimum cardinality, so that $CTATD(G) = 5$.

Observation 3.3. There exists a graph G for which, $\gamma(G) = \gamma_{[1,2]cc} = CTATD(G)$.

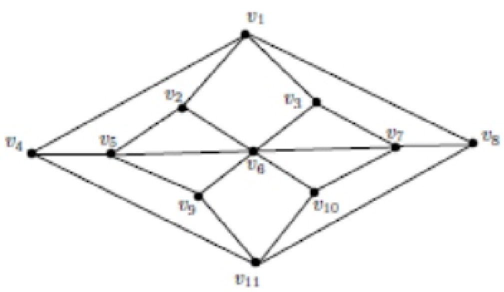

Figure 3.4.

In figure 3.4, $S = \{v_4, v_6, v_8\}$ is a dominating set, $[1, 2]$-complementary connected dominating set and complementary triple connected at most twin dominating set of smallest size. Hence $\gamma(G) = \gamma_{[1,2]cc} = CTATD(G)$.

Theorem 3.1. *For a connected graph G with* $p \geq 3, \left\lceil \frac{p}{\Delta+1} \right\rceil \leq CTATD(G)$ *and the bounds is sharp.*

Proof. Since, $\left\lceil \frac{p}{\Delta+1} \right\rceil \leq \gamma(G)$ and $\gamma(G) \leq CTATD(G)$ and the result follows. □

Example 3.1.

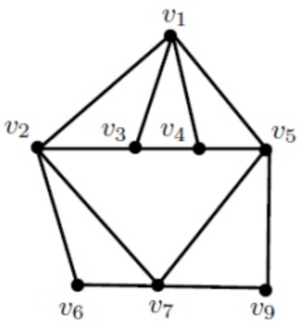

Figure 3.5.

In figure 3.5, $p = 8$ and $\Delta = 4$. Hence $S = \{v_1, v_7\}$ is a CTATD set of minimum cardinality. Hence $CTATD(G) = \left\lceil \frac{p}{\Delta+1} \right\rceil = \left\lceil \frac{8}{4+1} \right\rceil = 2.$

4. Exact Value of $CTATD(G)$-Number for Some Standard Graphs

(1) $CTATD(W_{1,n}) = 1.$

(2) $CTATD(K_p) = 1.$

(3) $CTATD(H_n) = n + 1.$

(4) $CTATD(K_{r,s}) = 2, r \geq s \geq 2.$

(5) $CTATD(TS_p) = p - 1.$

(6) For a path P_p, $p \geq 3$, $CTATD(D_2(P_p)) = p.$

(7) For a cycle C_p, $p \geq 3$, $CTATD(D_2(C_p)) = p.$

(8) $CTATD(Fl_p) = p - 1.$

(9) $CTATD(K_p \cup K_p + e) = 2, p \geq 3.$

5. Complementary Triple Connected at Most Twin Domination Number for Peculiar Types of Graphs

Observation 5.1.

(1) $CTATD(M_d(P_p)) = p,\ p \geq 4.$

(2) $CTATD(M_d(C_p)) = p,\ p \geq 4.$

(3) $CTATD(C(P_p)) = p-1,\ p \geq 3.$

(4) $CTATD(C(C_p)) = p,\ p \geq 3.$

Theorem 5.1. *For a path* $P_p,\ p \geq 3,\ CTATD(T(P_p)) = \left\lceil \frac{2p-1}{5} \right\rceil.$

Proof. Let $P_p = (v_1, v_2, \ldots, v_p),\ p \geq 3.$ This gives $V(T(P_p)) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_{p-1}\},\ E(T(P_p)) = \{v_iu_i,\ v_iv_{i+1},\ u_iv_{i+1},\ u_ju_{j+1}; 1 \leq i \leq p-1; 1 \leq i \leq p-2\}.$ Let $S_1 = \{v_i, u_j : i \equiv 2(\text{mod } 5); j \equiv 4(\text{mod } 5)\}.$ Assume

$$S = \begin{cases} S_1 & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3(\text{mod} 5) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 1 \text{ or } 4(\text{mod} 5). \end{cases}$$ Then S is a CTATD-set of $T(P_p)$

and hence $CTATD(T(P_p)) \leq |S| = \left\lceil \frac{2p-1}{5} \right\rceil$. Let S' be a CTATD-set of $T(P_p)$. Since any set D of cardinality at most $k = \left\lceil \frac{2p-1}{5} \right\rceil - 1$ is not a dominating set. We have $|S'| \geq k+1 = \left\lceil \frac{2p-1}{5} \right\rceil$. Hence the result follows. □

Theorem 5.2. *For a cycle* $C_p,\ p \geq 3,\ CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil.$

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_1),\ p \geq 3.$ This gives $V(T(C_p)) = \{v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, u_{p-1}\},\ E(T(C_p)) = \{v_iu_i,\ v_iv_{i+1},\ u_iv_{i+1},\ u_ju_{j+1},\ u_iu_{i+1},\ v_pv_1,\ u_pv_1; 1 \leq i \leq p-1\}.$

Let $S_1 = \{v_i, u_j : i \equiv 2(\text{mod } 5); j \equiv 4(\text{mod } 5)\}.$

Assume $$S = \begin{cases} S_1 & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3(\text{mod} 5) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 1 \text{ or } 3(\text{mod} 5). \end{cases}$$

Then S is a CTATD-set of $T(C_p)$ and hence $CTATD(T(C_p)) \leq |S| = \left\lceil \frac{2p}{5} \right\rceil$. Let S' be a CTATD-set of $T(C_p)$. Since any set D of cardinality at most $k = \left\lceil \frac{2p}{5} \right\rceil - 1$ is not a dominating set, we have $|S'| \geq k + 1 = \left\lceil \frac{2p}{5} \right\rceil$.

Hence the result follows. □

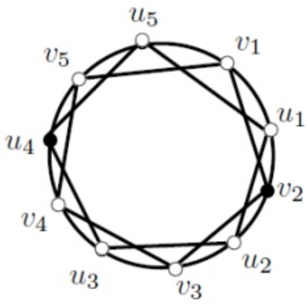

Figure 5.6.

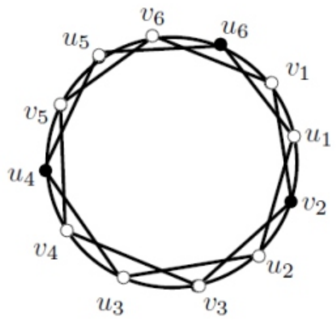

Figure 5.7.

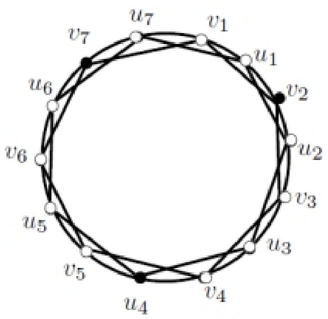

Figure 5.8.

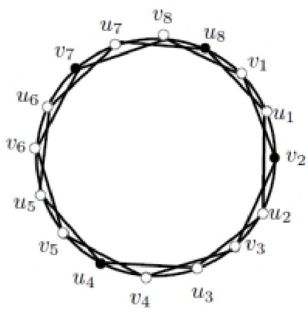

Figure 5.9.

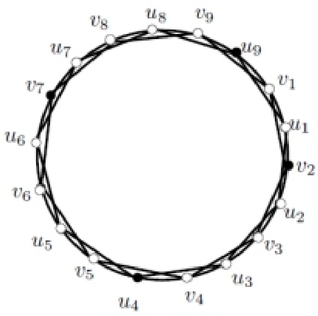

Figure 5.10.

Demonstration

Here, darked vertices are the CTATD-set, which is our S

In figure 5.6, $|S| = 2$.

As $p = 5$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_5)) = \left\lceil \frac{2 \times 5}{5} \right\rceil = 2$.

In figure 5.7, $|S| = 3$.

As $p = 7$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_7)) = \left\lceil \frac{2 \times 7}{5} \right\rceil = 3.$

In figure 5.9, $|S| = 4.$

As $p = 8$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_8)) = \left\lceil \frac{2 \times 8}{5} \right\rceil = 4.$

In figure 5.10, $|S| = 4.$

As $p = 9$, $CTATD(T(C_p)) = \left\lceil \frac{2p}{5} \right\rceil$ implies $CTATD(T(C_9)) = \left\lceil \frac{2 \times 9}{5} \right\rceil = 4.$

Theorem 5.3. *For a path* P_p, $p \geq 3$, $CTATD(M_r(P_p)) =$

$$\begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor + 1 & \text{if } p \equiv 0 \text{ or } 1 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 2 \text{ or } 3 (\text{mod } 4). \end{cases}$$

Proof. Let $P_p = (v_1, v_2, \ldots, v_p)$ and let the copies of $p'_p = (u_1, u_2, \ldots, u_p).$

This gives $E(M_r(P_p)) = \{u_i v_i, u_j u_{j+1}, v_j v_{j+1} : 1 \leq i \leq p; 1 \leq j \leq p-1\}.$

Let $S_1 = \{v_i, u_j : i \equiv 1(\text{mod } 4); j \equiv 3(\text{mod } 4)\}.$

$$\text{Assume } S = \begin{cases} S_1 \cup \{v_p\} & \text{if } p \equiv 0(\text{mod } 4) \\ S_1 & \text{if } p \equiv 1 \text{ or } 3(\text{mod } 4) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 0(\text{mod } 4) \end{cases}$$

Then S is a CTATD-set of $M_r(P_p)$ and hence

$$CTATD(M_r(P_p)) \leq |S| = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor + 1 & \text{if } p \equiv 0 \text{ or } 1(\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 2 \text{ or } 3(\text{mod } 4) \end{cases}$$

Let S' be a CTATD-set of $M_r(P_p)$. Since $D \subseteq V$ such that

$$|D| \leq k = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor & \text{if } p \equiv 0 \text{ or } 1(\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil - 1 & \text{if } p \equiv 2 \text{ or } 3(\text{mod } 4) \end{cases}$$

is not a dominating set, we have

$$| S' | \geq k+1 = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor + 1 & \text{if } p \equiv 0 \text{ or } 1 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 2 \text{ or } 3 (\text{mod } 4). \end{cases}$$

Hence the result follows. □

Theorem 5.4. *For a cycle* C_p, $p \geq 3$, $CTATD(M_r(C_p)) =$
$$\begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil + 1 & \text{if } p \equiv 1 (\text{mod } 4). \end{cases}$$

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$ and let the copies of $C'_p = (u_1, u_2, \ldots, u_p, u_1)$. This gives $E(M_r(C_p)) = \{u_i v_i, u_j u_{j+1}, v_j v_{j+1}, v_p v_1, u_p u_1 : 1 \leq i \leq p; 1 \leq j \leq p-1\}$. Let $S_1 = \{v_i, u_j : i \equiv 1 (\text{mod } 4); j \equiv 3 (\text{mod } 4)\}$.

Assume $S = \begin{cases} S_1 & \text{if } p \equiv 0 \text{ or } 1 \text{ or } 3 (\text{mod } 4) \\ S_1 \cup \{v_p\} & \text{if } p \equiv 2 (\text{mod } 4). \end{cases}$

Then S is a CTATD-set of $M_r(C_p)$ and hence

$$CTATD(M_r(C_p)) \leq | S | = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil + 1 & \text{if } p \equiv 1 (\text{mod } 4) \end{cases}$$

Let S' be a CTATD-set of $M_r(C_p)$. Since $D \subseteq V$

$$| D | \leq k = \begin{cases} 2\left\lfloor \frac{p}{4} \right\rfloor - 1 & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\text{mod } 4) \\ 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 1 (\text{mod } 4) \end{cases}$$

is not a dominating set, we have

$$| S' | \geq k+1 = \begin{cases} 2\left\lceil \frac{p}{4} \right\rceil & \text{if } p \equiv 0 \text{ or } 2 \text{ or } 3 (\text{mod } 4) \\ 2\left\lfloor \frac{p}{4} \right\rfloor & \text{if } p \equiv 1 (\text{mod } 4). \end{cases}$$

Hence the result follows.

Observation 5.2.

(1) $CTATD(M_r(K_p)) = 2.$

(2) $CTATD(M_r(W_{1,n})) = 2.$

Theorem 5.5. *For a closed Helm graph* CH_n, *for* $n \geq 3$ *then,* $CTATD(CH_n) = \left\lceil \frac{n}{3} \right\rceil + 1.$

Proof. Let v_0 be apex vertex of the closed Helm graph CH_n, $(v_1, v_2, \ldots, v_n)$ be the inner cycle of CH_n and $(v'_1, v'_2, \ldots, v'_n)$ be the outer cycle of CH_n. Let $S = \{v'_i : i \equiv 1(\text{mod}3)\} \cup \{v_0\}$. Then S is a CTATD-set of (CH_n) and hence $CTATD(CH_n) \leq |S'| = \left\lceil \frac{n}{3} \right\rceil + 1$. Let S' be a CTATD-set of (CH_n). Since any set D of cardinality at most $k = \left\lceil \frac{n}{3} \right\rceil$ is not dominating set, we have $|S'| \geq k + 1 = \left\lceil \frac{n}{3} \right\rceil + 1$.

Hence the result follows. □

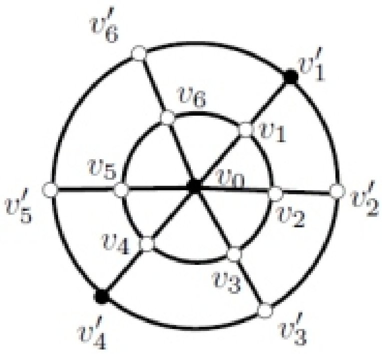

Figure 5.19.

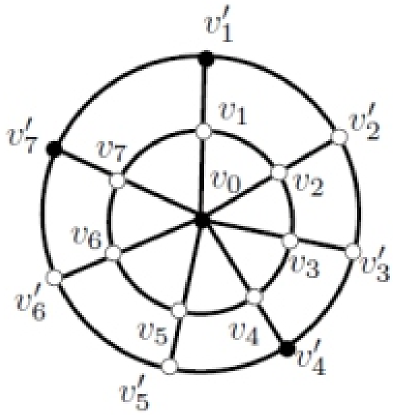

Figure 5.20.

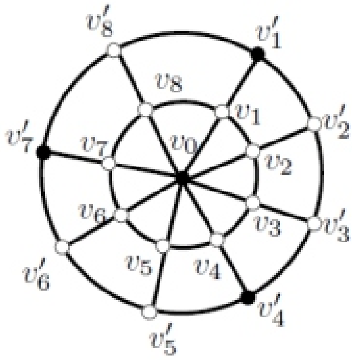

Figure 5.21.

Demonstration

Here, darked vertices are the CTATD-set, which is our S

In figure 5.19, $|S| = 3$.

As $p = 6$, $CTATD(CH_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ implies $CTATD(CH_6) = \left\lceil \frac{6}{4} \right\rceil + 1 = 3$.

In figure 5.20, $|S| = 4$.

As $p = 7$, $CTATD(CH_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ implies $CTATD(CH_7) = \left\lceil \frac{7}{4} \right\rceil + 1 = 4$.

In figure 5.21, $|S| = 4$.

As $p = 8$, $CTATD(CH_n) = \left\lceil \frac{n}{4} \right\rceil + 1$ implies $CTATD(CH_8) = \left\lceil \frac{8}{4} \right\rceil + 1 = 4$.

6. Conclusion

In this article we developed a new parameter called CTATD-number and found its exact values for some special types of graphs such as complete graph, wheel graph, Helm graph etc. The authors obtained results for various types of product graphs like Cartesian product, corona product, lexicographic product, strong product, which will be investigated in the subsequent articles.

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MORE RESULTS ON DEGREE PARTITION NUMBER

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ABSTRACT

The whole world is currently dealing with a major problem caused by Covid 19, which necessitates social separation in many aspects. In certain circumstances, a need may emerge in which a certain group of individuals or components must be divided into multiple groups in order to meet certain requirements. We define a vertex partition $\pi_k = \{V_1, V_2, \ldots, V_k\}$ *on the vertex set V of a graph G which is said to be a similar degree partition if the sum of degrees of vertices in each class* $V_i, 1 \le i \le k,$ *differs from that of other by at most 1. The degree partition number of* $G, \psi_D(G) = \max\{k/\pi_k$ *is a similar degree partition of G.In this paper we present the degree partition number of some graphs and we establish some bounds for this parameter.*

Keywords: graph partition, partitioning, degree partition number, degree partitioning.

1. INTRODUCTION

Only finite, simple, undirected graphs are considered in this study. For basic notations and terminology that are not included here, [1, 2] can be used to look up. The degree set of a graph is indicated by *D(G)*,while the degree of a vertex v is denoted by deg vor d(v). In a graph, the minimum and maximum degree of vertices are represented by *deg v* or *d(v)* respectively.

If every vertex of a graph G has degree r, the graph is said to be r-regular. A graph with n vertices is complete graph if it is *(n − 1)*- regular. The graph is known to be *(r, r + 1)* -biregular if any vertex of *G* is of degree either *r or r +1.* A graph *G(V, E)* is connected if there exists a path connecting every two vertices of *G*. Path and cycle on n vertices are denoted by Pnand *Cn* respectively.

A graph *G(V, E)* is called a bipartite graph with bipartition (*V', V''*)if any edge *uv "* Ehas its one of its ends in *V''* and other in *V''*.If every vertex in *V''* is adjacent to every other vertex in *V ",* such bipartite graph is called complete bipartite graph denoted by *Km,n* where *V1 = m* and *V2 = n.*

The Cartesian product graph *G = G1 G2* of two graphs *G1*and *G2*with disjoint vertex sets *V1*and *V2* and edge set E1and E2is the graph with vertex set *V1 = V2*and the vertex *u1, u2*, uis adjacent to the vertex ( ) 1 2 v = v , vif u1 = v1and u2is adjacent to v2or u2 = v2andu1is adjacent to . v1 Grid graph *G(m, n)* is a cartesian product of two paths Pmand Pn.The ladder graph can be obtained as the Cartesian product of two path graphs P2and Pn.The friendship graphFncan be created by linking ncopies of the cycle graph C3 with a common vertex.

In many cases, a situation may arise where a single group of people or components must be separated into various groups in order to meet special needs. Graph models are one of the techniques to depict any system. To investigate the nature and properties of a network, our mathematicians devise a variety of partitioning methods..

Graph partition is the process of reducing a graph to smaller graphs by partitioning its vertex set into mutually incompatible groups. There are numerous research concepts in the literature that are based on partitioning the vertex and edge sets of a graph.

The general chromatic partition, bilinear partitions, trilinear partitions are some examples of graph partitions that can be referred from [3, 4].

This study was prepared during the Corona virus pandemic, which necessitates social separation in all aspects. Every system, however, must be dynamic for the country's economic and educational well-being. To meet the need of the hour, the system must be subdivided into smaller groups with more or less identical capacity. This serves as the foundation for the investigation of degree partition number [5, 6], which is presented in this work.

Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$, $(k \geq 2)$ be a partition of the vertex set $V(G)$. π_k is called a similar degree partition if the sum of degrees of vertices in any class of π_k differs from that of other by at most 1. i.e., if $|\sum_{v \in V_i} d(v) - \sum_{v \in V_j} d(v)| \leq 1$ for $1 \leq i, j \leq k$. When this difference equals zero for any two classes of a partition π_k, then it is called perfect similar degree partition. The degree partition number of a graph $\psi_D(G)$ is defined as $\max\{k/\pi_k$ is a similar degree partition of $G\}$ and such π_k is called the maximal similar degree partition.

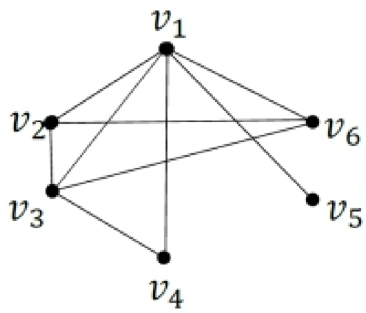

Figure 1.

Here $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1\}$, $V_2 = \{v_2, v_3\}$, $V_3 = \{v_4, v_5, v_6\}$

$\pi_2 = \{V_1, V_2\}$ where $V_1 = \{v_1, v_5, v_6\}$, $V_2 = \{v_2, v_3, v_4\}$

$\pi_3' = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1, v_5\}$, $V_2 = \{v_2, v_6\}$, $V_3 = \{v_3, v_4\}$.

One can confirm that π_3 is not a similar degree partition. π_2 and π_3' are similar degree partitions. As no other similar degree partition π_k, $k \geq 4$ exists for this graph, $\psi_D(G) = 3$ and π_3' is a maximal similar degree partition of the graph that we considered above.

In this section, we present some basic results and bounds on the degree partition number of a graph.

Fact 1. The degree partition number of any grid graph is $|V| - 2$.

Proof. Grid graph $G(m, n)$ is a graph with mn vertices. Let $V = \{v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}\}$.

We may note that,

$$\deg v_{ij} = \begin{cases} 2 & \text{if } (i, j) = (1, 1), (1, n), (m, 1), (m, n) \\ & \text{if } (i, j) = (1, 2), (1, 3), \ldots, (1, n-1), \\ 3 & (m, 2), (m, 3), \ldots, (m, n-1), \\ & (2, 1), (3, 1), \ldots, (m-1, 1), \\ & (2, n), (3, n), \ldots, (m-1, n) \\ 4 & \text{otherwise} \end{cases}$$

By taking corner vertices in pair, and remaining vertices as individual classes we get the required similar degree partition. So, $\psi_D(G(m, n)) = mn - 2$. □

Fact 2. The degree partition number of any friendship graph is 3.

Proof. Let the central vertex be denoted by v and the set of remaining vertices be $\{v_1, v_2, \ldots, v_{2n}\}$.

Then $\deg v = 2n$ and $\deg v_i = 2$ for all $i = 1, 2, \ldots, 2n$. The partition $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{v\}, V_2 = \{v_1, v_2, \ldots, v_n\}$ and $V_3 = \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ is the required similar degree partition. Hence $\psi_D(G) = 3$. □

Fact 3. The degree partition number of bipartite graph G is at least 2.

In fact, we can note that the bipartition of the vertex set itself forms a degree partition of G. □

Fact 4. The degree partition number of complete graph, path, cycle, Peterson graph, $(n, n+1)$-complete bipartite graph and ladder graph is $|V(G)|$. □

In fact, we can state the following theorem.

Theorem 5. $\psi_D(G) = |V(G)|$ *if and only if G is either a regular graph or* $(n, n+1)$*-biregular graph.* □

Theorem 6. $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$.

Proof. Let G be a graph with n vertices. Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$ be a maximal similar degree partition of G. Then $\psi_D(G) = k$.

Clearly, there exists at least one partition say V_1 such that $\sum_{v_i \in V_1} \deg v_i \geq \Delta$.

Also, $\sum_{v_i \in V_j} \deg v_i \geq \Delta - 1$ for $j = 2, 3, \ldots, k$.

Adding the above k inequalities, we get

$$\sum_{v_i \in V(G)} \deg v_i \geq \Delta + (k-1)(\Delta - 1).$$

$$\therefore k - 1 \leq \frac{\sum_{v_i \in V_1} \deg v_i - \Delta}{\Delta - 1} \Rightarrow k \leq \frac{\sum_{v_i \in V_1} \deg v_i - \Delta}{\Delta - 1} + 1$$

$$\Rightarrow k \leq \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1}$$

Hence, $k \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$ since k is an integer.

Always $k \geq 1$.

Thus $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$.

Corollary 7. *If* $\psi_D(G) = \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i - 1}{\Delta - 1} \right\rfloor$, *then there exists at least one partition class say* V_i *in* ψ_D *such that* V_i *contains max-degree vertex alone.* □

Theorem 8. *If degree of each vertex of* G *is even, then* $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta} \right\rfloor$.

Proof. Let G be a graph with n vertices and degree of each vertex be even.

Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$ be a maximal similar degree partition of G. Then $\psi_D(G) = k$.

Since degree of each vertex is even, π_k should be a perfect similar degree partition of G.

Then, $\sum_{v_i \in V_j} \deg v_i \geq \Delta$ for all $j = 1, 2, 3, \ldots, k$.

Adding the above k inequalities, we get

$$\sum_{v_i \in V(G)} \deg v_i \geq k\Delta.$$

$\therefore k \leq \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta}$. Hence, $k \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta} \right\rfloor$ since k is an integer.

Always $k \geq 1$. Thus $1 \leq \psi_D(G) \leq \left\lfloor \frac{\sum_{v_i \in V_1} \deg v_i}{\Delta} \right\rfloor$. □

Theorem 9. *For* $n \geq 4$, $\psi_D(K_{2,n}) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 3 & \text{if } n \equiv \pm 1 (\text{mod } 6) \\ 2 & \text{if } n \equiv 3 (\text{mod } 6). \end{cases}$

Proof. Let $K_{2,n}$ be a complete bipartite graph with bipartition (V', V'')

where $V' = \{u_1, u_2\}$, $V'' = \{v_1, v_2, \ldots, v_n\}$. Here $\deg u_i = n$ for $i = 1, 2$ and $\deg v_i = 2$ for $i = 1, 2, \ldots, n$.

Also, $\sum_{v_i \in V(K_{2,n})} \deg v_i = 2(2)(n) = 4n$.

By theorem 6, no matter whether n is odd or even, $\psi_D(K_{2,n}) \leq 4$.

Also, since it is bipartite, $\psi_D(K_{2,n}) \geq 2$. So, $2 \leq \psi_D(K_{2,n}) \leq 4$.

Let $\pi_4 = \{V_1, V_2, V_3, V_4\}$ be a similar degree partition of $K_{2,n}$, then $V_1 = \{u_1\}$, $V_2 = \{u_2\}$.

Now since $\deg v_i = 2$ for $1 \leq i \leq n$ and $\sum_{v_i \in V''} \deg v_i = 2n$, we need to partition V'' into V_3 and V_4 so that $\sum_{v_i \in V_j} \deg v_i = n$ for $j = 3, 4$.

This is possible only when n is even.

Hence, $\psi_D(K_{2,n}) = 4$ only when n is even, i.e., $n \equiv 0(\bmod 2)$ or $(n \equiv 0, 2, 4(\bmod 6))$.

For the remaining cases, if $\pi_3 = \{V_1, V_2, V_3\}$ forms a similar degree partition of $K_{2,n}$, then $V_1 = \{u_1, v_1, v_2, \ldots, v_k\}$, $V_2 = \{u_2, v_{k+1}, v_{k+2}, \ldots, v_{2k}\}$

and $V_3 = \{v_{2k+1}, v_{2k+2}, \ldots, v_n\}$ where n is odd.

Here, $\sum_{v_i \in V_j} \deg v_i = n + 2k$ for $j = 1, 2,$ and $\sum_{v_i \in V_3} \deg v_i$

$= 2(n - 2k)$

Note that $n + 2k$ and $2(n - 2k)$ are of different parity considering the fact that n is odd.

$$\therefore 2(n - 2k) = n + 2k \pm 1 \Rightarrow n = 6k \pm 1$$

Hence, $\psi_D(K_{2,n}) = 3$ if $n \equiv \pm 1(\bmod 6)$.

And $\pi_2 = \{V', V''\}$ forms a maximal similar degree partition in the remaining case $n \equiv 3(\bmod 6)$. □

As an illustration, $K_{2,5}$ is shown in Figure 2.

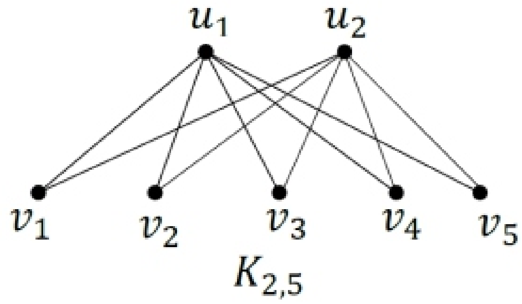

Figure 2.

Let $K_{2,5}$ have bipartition (V', V'') where $V' = \{u_1, u_2\}$ and $V'' = \{v_1, v_2, v_3, v_4, v_5\}$. The partition $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{u_1, v_1\}, V_2 = \{u_2, v_2\}$ and $V_3 = \{v_3, v_4, v_5\}$ stands as the maximal similar degree partition of $K_{2,5}$. □

Theorem 10. $\psi_D(K_{3k+2, 6k+5}) = 4k + 3$ *for* $k \geq 1$.

Proof. Let $K_{3k+2, 6k+5}$ be a complete bipartite graph with bipartition (V', V'') where $V' = \{u_1, u_2, \ldots, u_{3k+2}\}, V'' = \{v_1, v_2, \ldots, v_{6k+5}\}$.

Here $\deg u_i = 6k + 5$ for $i = 1, 2, \ldots, 3k + 2$ and $\deg v_i = 3k + 2$ for $i = 1, 2, \ldots, 6k + 5$.

$\pi_{4k+3} = \{V_1, V_2, \ldots, V_{4k+3}\}$ where $V_1 = \{u_1, v_1\}, V_2 = \{u_2, v_2\}, \ldots, V_{3k+2}$

$= \{u_{3k+2}, v_{3k+2}\}, V_{3k+3} = \{v_{3k+3}, v_{3k+4}, v_{3k+5}\}, \ldots, V_{4k+3} = \{v_{6k+3}, v_{6k+4}, v_{6k+5}\}$ forms a maximal similar degree partition of $K_{3k+2, 6k+5}$. It can be easily verified that the degree sum of the partition classes $V_1, V_2, \ldots, V_{3k+2}$ are $9k + 7$ and the degree sum of the partition classes $V_{3k+3}, V_{3k+4}, \ldots, V_{4k+3}$ are $9k + 6$.

$$\therefore \psi_D(K_{3k+2, 6k+5}) = 4k + 3.$$ □

For example, let the bipartition (V', V'') of $K_{5,11}$ be given by $V' = \{u_1, u_2, u_3, u_4, u_5\}$ and $V'' = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$.

The partition $\pi_7 = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7\}$ where $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$, $V_3 = \{u_3, v_3\}$, $V_4 = \{u_4, v_4\}$, $V_5 = \{u_5, v_5\}$, $V_6 = \{v_6, v_7, v_8\}$ and $V_7 = \{v_9, v_{10}, v_{11}\}$ serves as the maximal similar degree partition of $K_{5,11}$ with degree sum as 16 for the partition classes V_1, V_2, V_3, V_4, V_5 and as 15 for the partition classes V_6 and V_7. □

Theorem 11. $\psi_D(K_{3k+1, 6k+1}) = 4k + 1$ *for* $k \geq 1$.

Proof. Let $K_{3k+1, 6k+1}$ be a complete bipartite graph with bipartition (V', V'') where $V' = \{u_1, u_2, \ldots, u_{3k+1}\}$, $V'' = \{v_1, v_2, \ldots, v_{6k+1}\}$.

Here $\deg u_i = 6k + 1$ for $i = 1, 2, \ldots, 3k + 1$ and $\deg v_i = 3k + 1$ for $i = 1, 2, \ldots, 6k + 1$.

$\pi_{4k+1} = \{V_1, V_2, \ldots, V_{4k+1}\}$ where $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}, \ldots, V_{3k+1} = \{u_{3k+1}, v_{3k+1}\}$, $V_{3k+2} = \{v_{3k+2}, v_{3k+3}, v_{3k+4}\}, \ldots, V_{4k+1} = \{v_{6k-1}, v_{6k}, v_{6k+1}\}$ forms a maximal similar degree partition of $K_{3k+1, 6k+1}$. One can verify that the degree sum of the partition classes $V_1, V_2, \ldots, V_{3k+1}$ are $9k + 2$ and the degree sum of the partition classes $V_{3k+2}, V_{3k+3}, \ldots, V_{4k+1}$ are $9k + 3$.

$$\therefore \psi_D(K_{3k+1, 6k+1}) = 4k + 1. \qquad \square$$

For instance, we consider $K_{7,13}$ with the bipartition (V', V'') where $V' = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and $V'' = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$.

The partition $\pi_9 = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9\}$ where $V_1 = \{u_1, v_1\}$, $V_2 = \{u_2, v_2\}$, $V_3 = \{u_3, v_3\}$, $V_4 = \{u_4, v_4\}$, $V_5 = \{u_5, v_5\}$, $V_6 = \{u_6, v_6\}$, $V_7 = \{u_7, v_7\}$, $V_8 = \{v_8, v_9, v_{10}\}$ and $V_9 = \{v_{11}, v_{12}, v_{13}\}$ forms the maximal similar degree partition of $K_{7,13}$ with degree sum as 20 for the partition classes $V_1, V_2, V_3, V_4, V_5, V_6, V_7$ and as 21 for the partition classes V_8 and V_9. □

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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we will discuss a unified approach to study the existence and uniqueness of solution of boundary value problems of fractional order subjected to non-local conditions. By the reduction of the problem to operator equation we establish the existence and uniqueness of solution. The approach used for the (Alpha -1) order nonlinear functional fractional differential equation can be applied to functional differential equations of any fractional orders.

Keywords: fractional derivative and integral, fractional differential equation, existence and uniqueness of solutions.

1. INTRODUCTION

Some results on the problem of existence and uniqueness of solution of differential equations of fractional order have been discussed by some authorswhich can be found in [1, 2]. The purpose of this paper is to discuss a newapproach to functional fractional differential equations, moreover this approach can be applied to functional differential equations of any fractional orders with nonlinear terms containing derivatives.

But for simplicity now we consider the following functional fractional differential equations of the form

$$\vartheta^{\alpha+2} = \phi(\eta,\ \vartheta(\eta),\ \vartheta(\varphi(\eta))),\ \eta \in [0,\ \sigma] \tag{1.1}$$

subjected to the general boundary conditions

$$\begin{aligned} B_1[\vartheta] &= \alpha_1\vartheta(0) + \beta_1\vartheta^{\alpha}(0) + \gamma_1\vartheta^{\alpha-1}(0) = b_1, \\ B_2[\vartheta] &= \alpha_2\vartheta(0) + \beta_2\vartheta^{\alpha}(0) + \gamma_2\vartheta^{\alpha-1}(0) = b_2, \\ B_3[\vartheta] &= \alpha_3\vartheta(0) + \beta_3\vartheta^{\alpha}(0) + \gamma_3\vartheta^{\alpha-1}(0) = b_3, \end{aligned} \tag{1.2}$$

such that

$$Rank\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ \gamma_1 & \gamma_2 & 0 \\ 0 & 0 & \alpha_3 \\ 0 & 0 & \beta_3 \\ 0 & 0 & \gamma_3 \end{pmatrix} = 3$$

the function $\varphi(\eta)$ is assumed to be continuous and maps $[0, \sigma]$ into itself.

Definition 1.1 Riemann-Liouville definition[3, 4, 5]. For $\alpha \in [n-1, n)$ the α-derivative of f is

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n x}{dt^n} \int_a^{\alpha} \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

Definition 1.2. Caputo definition[3, 4, 5]. For $\alpha \in [n-1, n)$ the α- derivative of f is

$${}_a^C D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau.$$

2. Existence and Uniqueness of Solution

To solve the problem (1.1)-(1.2) we introduce the nonlinear operator Ω defined in the space of continuous functions $C[0, \sigma]$ by the formula:

$$(\Omega\psi)(\eta) = \phi(\eta, u(\eta), u(\varphi(\eta))), \tag{2.1}$$

where $u(\eta)$ is the solution of the problem

$$\vartheta^{\alpha+2}(\eta) = \psi(\eta), \ 0 < \eta < 1$$

$$B_1[\vartheta] = b_1,$$

$$B_2[\vartheta] = b_2,$$

$$B_3[\vartheta] = b_3, \tag{2.2}$$

where $B_1[\vartheta], B_2[\vartheta], B_3[\vartheta]$ are defined by (1.2).

Proposition 2.1. *Suppose the function ψ is a fixed point of the operator Ω, i.e., ψ is the solution of the operator equation*

$$\Omega\psi = \psi, \tag{2.3}$$

where Ω is defined by (2.1)-(2.2) *then the function $\vartheta(\eta)$ determined from the BVP* (2.2) *is a solution of the BVP* (1.1)-(1.2). *Conversely, suppose the function $\vartheta(x)$ is the solution of the BVP* (1.1)-(1.2) *then the function*

$$\psi(\eta) = \phi(\eta, \vartheta(\eta), \vartheta(\varphi(\eta)))$$

satisfies the operator equation (2.3).

Now, let $\Phi(\eta, s)$ be the Green function of the problem (2.2). Then the solution of the problem can be represented in the form

$$\vartheta(\eta) = g(\eta) + \frac{1}{\Gamma(-\alpha - n)} \int_0^{\sigma} \frac{\Phi^{(n)}(\eta, s)\psi(s)ds}{(\sigma - s)^{-\alpha+1-n}}, \tag{2.4}$$

where $g(\eta)$ is the polynomial of $\alpha + 1$ degree satisfying the boundary conditions

$$B_1[\vartheta] = b_1,\ B_2[\vartheta] = b_2,\ B_3[\vartheta] = b_3, \tag{2.5}$$

Denote

$$\mathcal{M}_0 = \max_{0 \le \eta \le \sigma} \frac{1}{\Gamma(-\alpha - n)} \int_0^1 \frac{|\ \Phi^{(n)}(\eta, s)\ |}{|\ (\sigma - s)^{-\alpha+1-n}\ |}. \tag{2.6}$$

For any positive number M define the domain

$$\mathcal{D}_{\mathcal{M}} = \{(\eta, \vartheta, v) |\ 0 \le \eta \le \sigma; |\ \vartheta\ | \le \|\ g\ \| + \mathcal{M}_0\mathcal{M}; |\ v\ | \le \|\ g\ \| + \mathcal{M}_0\mathcal{M}\}, \tag{2.7}$$

where $\|\ g\ \| = \max_{0 \le \eta \le \sigma} |\ g(\eta)\ |$.

As usual, we denote by $B[0, \mathcal{M}]$ the closed ball of the radius $\mathcal{M}$ centered at 0 in the space of continuous functions $C[0, \sigma]$.

Theorem 2.2. *Suppose that*:

(i) *The function* $\varphi(\eta)$ *is a continuous map from* $[0, \sigma]$ *to* $[0, \sigma]$.

(ii) *The function* $\phi^{(n)}(\eta, \vartheta, v)$ *is continuous and bounded by* $\mathcal{M}$ *in the domain* $D_{\mathcal{M}}$, *that is,*

$$|\ \phi^{(n)}(\eta, \vartheta, v)\ | \le \mathcal{M}\ \forall\ (\eta, \vartheta, v) \in D_{\mathcal{M}}. \tag{2.8}$$

(iii) *The function* $\phi^{(n)}(\eta, \vartheta, v)$ *satisfies the Lipschitz conditions in the variables* u, v *with the coefficients* $L_1, L_2 \ge 0$ *in* $D_{\mathcal{M}}$, *that is,*

$$|\ \phi^{(n)}(\eta, \vartheta_2, v_2) - \phi^{(n)}(\eta, \vartheta_1, v_1)\ | \le L_1 |\ \vartheta_2 - \vartheta_1\ | + L_2 |\ v_2 - v_1\ |$$

$$\forall\ (\eta, \vartheta_i, v_i) \in D_{\mathcal{M}} (i = 1, 2) \tag{2.9}$$

(iv) $q = (L_1 + L_2)\mathcal{M}_0 < 1.$ (2.10)

The problem (1.1)-(1.2) *has a unique solution* $\vartheta(\eta) \in C^3[0, \sigma]$, *satisfying*

$$|\vartheta(\eta)| \le \| g \| + \mathcal{M}_0 \mathcal{M} \forall \eta \in [0, \sigma]. \quad (2.11)$$

Proof. Claim 1. The operator Ω is a mapping $\mathcal{B}[0, \mathcal{M}] \to \mathcal{B}[0, \mathcal{M}]$.

Indeed, for any $\psi \in \mathcal{B}[0, \mathcal{M}]$, we have $\| \psi \| \le \mathcal{M}$. Let $\vartheta(\eta)$ be the solution of the problem (2.2). From (2.4) it follows

$$|\vartheta(\eta)| \le \| g \| + \mathcal{M}_0 \mathcal{M} \forall \eta \in [0, \sigma]. \quad (2.12)$$

Since $0 \le \varphi(\eta) \le a$, we have

$$|\vartheta(\varphi(\eta))| \le \| g \| + \mathcal{M}_0 \mathcal{M} \forall t \in [0, \sigma].$$

Therefore, if $\eta \in [0, \sigma]$ then $(\eta, \vartheta(\eta), \vartheta(\varphi(\eta))) \in D_{\mathcal{M}}$. By the supposition (2.8) we have $|\phi(\eta, \vartheta(\eta), \vartheta(\varphi(\eta)))| \le \mathcal{M} \forall \eta \in [0, \sigma]$. From (2.1) we have $|(\Omega\psi)(\eta)| \le \mathcal{M} \forall \eta \in [0, \sigma]$. It means $|(\Omega\psi)| \mathcal{M}$ or $\Omega\psi \in B[0, B]$.

Claim 2. Ω is a contraction in $\mathcal{B}[0, \mathcal{M}]$.

If $\psi_1, \psi_2 \in \mathcal{B}[0, \mathcal{B}]$ and $\vartheta_1(\eta), \vartheta_2(\eta)$ is the solutions of the problem (2.2), respectively. Then from the supposition (2.9) we obtain

$$|\Omega\psi_2 - \Omega\psi_1| \le L_1 |\vartheta_2(t) - \vartheta_1(\eta)| + L_2 |\vartheta_2(\varphi(\eta)) - \vartheta_1(\varphi(\eta))|. \quad (2.13)$$

From the representations

$$\vartheta_i(\eta) = g(\eta) + \frac{1}{\Gamma(-\alpha - n)} \int_0^{\sigma} \frac{\Phi^{(n)}(\eta, s)\psi_i(s)ds}{(\sigma - s)^{-\alpha+1-n}}, (i = 1, 2) \quad (2.14)$$

and (2.6) we have

$$|\vartheta_2(\eta) - \vartheta_1(\eta)| \le \mathcal{M}_0 \| \psi_2 - \psi_1 \|,$$

$$|\vartheta_2(\varphi(\eta)) - \vartheta_1(\varphi(\eta))| \le \mathcal{M}_0 \| \psi_2 - \psi_1 \|$$

Together with the above estimates and (2.13), from the supposition (2.10) we get

$$\| \Omega\psi_2 - \Omega\psi_1 \| \le q \| \psi_2 - \psi_1 \|, q < 1.$$

Together with the above estimates and (2.13), from the supposition (2.10) we get

$$\| \Omega\psi_2 - \Omega\psi_1 \| \leq q \| \psi_2 - \psi_1 \|, q < 1.$$

Thus, Ω is a contraction mapping in $\mathcal{B}[0, \mathcal{M}]$.

Therefore, the operator equation (2.3) has a unique solution $\psi \in \mathcal{B}[0, \mathcal{M}]$. From Proposition 2.1 the solution of the problem (2.2) for this right-hand side $\psi(\eta)$ is the solution of the original problem (1.1)-(1.2).

3. Conclusion

In this paper, we have discussed the existence and uniqueness of solution of boundary value problems of fractional order subjected to non-local conditions. By the reduction of the problem to operator equation we established the existence and uniqueness of solution. The approach used for the α - 1 order nonlinear functional fractional differential equation can be applied to functional differential equations of any fractional orders. It also can be applied to fractional order integro-differential equations.

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COVER EDGE PEBBLING NUMBER FOR JAHANGIR GRAPHS J1,m, J2,m, J3,m, J4,m AND J5,m

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ABSTRACT

Let G be a connected graph. An edge pebbling move on G is the process of removing two pebbles from one edge and placing one pebble on the adjacent edge. The cover edge pebbling number of G, denoted by CPE (G) is the minimum number of pebbles required to place a pebble on all the edges of G, however might be the initial configuration is. In this paper, we determine the cover edge pebbling number for Jahangir graphs J1,m, J2,m, J3,m, J4,m and . J5,m.

Keywords: cover edge pebbling number, Jahangir graph.

1. INTRODUCTION

least number of pebbles needed in a graph G so that we can move a pebble to any arbitrary target vertex by a sequence of pebbling move whatever might be the initial configuration is.

The concept of cover pebbling was first introduced by Crull [2]. The cover pebbling number *CP(G)* is the least number of pebbles needed in a graph *G* so that we can move one pebble to all the vertices of the graph G.

In [6] a new concept namely edge pebbling number and cover edge pebbling number has been introduced and cover edge pebbling number for certain standard graphs namely path, complete graph, friendship graph and star graph have been determined.

In edge pebbling, pebbles will be distributed on the edges of the graph instead of the vertices. An edge pebbling move is the process of removing two pebbles from one edge and placing one pebble on the adjacent edge. Edge pebbling number *PE(G)*is the minimum number of pebbles needed in a graph G to reach any arbitrary target edge by a sequence of edge pebbling move regardless of initial configuration of pebbles. The cover edge pebbling number *CPE (G)* is the least number of pebbles needed in a graph G so that we can move one pebble to all the edges of the graph *G.* In this paper we establish the cover edge pebbling number for certain classes of Jahangir graph.

2. Cover Edge Pebbling Number

Definition 2.1[6]. A cover edge pebbling number *CPE(G)* of a graph *G* is defined as, however the pebbles are initially placed in the edges, the minimum number of pebbles required to place a pebble in all the edges.

Definition 2.2[6]. The distance between two edges x and y is defined as, $d(x, y) = d(v_i, v_j) - 2$ where $x = v_i v_{i+1}$, $y = v_{j-1} v_j$ and $d(v_i, v_j)$ is the length of the shortest path between v_i and v_j.

Definition 2.3[6]. The distance $d(x)$ of an edge x in a graph G is the sum of the distances from x to each other edge of $E(G)$, where $E(G)$ is the edge set of G.

(i.e.) $d(x) = \sum_{x \in E(G)} d(y, x) \; \forall \; y \in E(G), \, x \neq y.$

Definition 2.4[6]. Let $x \in E(G)$, then x is called a key edge if $d(x)$ is a maximum.

Result [6]. After finding the key edge of a graph find the minimum number of pebbles to be placed on that key edge such that a pebble is placed on each of the edges of the given graph. Then the minimum number of pebbles placed on the key edge is the cover edge pebbling number. Using this result the following theorems are proved.

Definition 2.5[5]. Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph on $nm + 1$ vertices, that is, a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

Note. In this paper, one additional vertex which is at the center is mentioned as central vertex.

Theorem 2.6. *For* $J_{1,m}(m = 3, 4)$, $CP_E(J_{1,m}(m = 3, 4)) = 8m - 11$.

Proof. $J_{1,m}(m = 3, 4)$ has $m + 1$ vertices and $2m$ edges. For $J_{1,3}$ all the edges are key edges. For $J_{1,4}$ the edges which lies on the cycle C_m are key edges. Choose any one of the key edges from $J_{1,m}(m = 3, 4)$. Let it be e_1. Each of the key edges are adjacent with 4 edges (i.e.) e_1 is adjacent with 4 edges. To place one pebble on each of these 4 edges, $4 * 2 = 8$ pebbles are needed in e_1 because of adjacency. There are remaining $2m - 5$ edges. On finding the shortest path these $2m - 5$ edges can be reached by crossing exactly one edge. Therefore $2^2(2m - 5)$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{1,m}(m = 3, 4)$ is

$$= 8 + 2^2(2m - 5) + 1$$

$$= 8m - 11.$$

Theorem 2.7. *For* $J_{1,m}(m \geq 5)$, $CP_E(J_{1,m}(m \geq 5)) = 12m - 31.$

Proof. $J_{1,m}(m \geq 5)$ has $m+1$ vertices and $2m$ edges. For $J_{1,m}(m \geq 5)$ the edges which lies on the cycle C_m are key edges. Choose any one of the key edges. Let it be e_1. e_1 is adjacent with 4 edges. ((i.e.) 2 edges from the cycle C_m and 2 edges which are incident with the central vertex). To place one pebble on each of these 4 edges, $4 * 2 = 8$ pebbles are needed in e_1 because of adjacency. There are remaining $m - 3$ edges from C_m and $m - 2$ edges which are incident with the central vertex. Among these $m - 3$ edges from C_m, 2 edges can be reached by crossing exactly one edge on finding the shortest path. Therefore $2(2^2)$ pebbles are needed in e_1. The remaining $m - 5$ edges can be reached by crossing exactly 2 edges on finding the shortest path. Therefore $2^3(m - 5)$ pebbles are needed in e_1. To reach the $m - 2$ edges which are incident with the central vertex one edge has to be crossed. Here in this case $2^3(m - 2)$ pebbles are needed in e_1. Altogether, $8 + 2(2^2) + 2^3(m - 5) + 2^2(m - 2)$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{1,m}(m \geq 5)$

Theorem 2.8. *For* $J_{2,m}(m \geq 3)$, $CP_E(J_{2,m}) = 20m - 29.$

Proof. $J_{2,m}(m \geq 3)$ has $2m+1$ vertices and $3m$ edges. For $J_{2,m}(m \geq 3)$ the edges which lies on the cycle C_{2m} are key edges. Choose any one of the key edges. Let it be e_1. Now, e_1 is exactly adjacent with 3 edges. (i.e.) 2 edges from the cycle C_{2m} and 1 edge from the edges which are incident with the central vertex. To place one pebble on each of these 3 edges, $3 * 2 = 6$ pebbles are needed in e_1. There are remaining $2m-3$ edges from the cycle C_{2m} and $m-1$ edges which are incident with the central vertex. Among these $2m-3$ edges from the cycle C_{2m}, 2 edges can be reached by crossing exactly one edge on finding the shortest path. Therefore $2(2^2)$ pebbles are needed in e_1. Remaining $2m-5$ edges from the cycle C_{2m} can be reached by crossing exactly 2 edges on finding the shortest path. Therefore $2^3(2m-5)$ pebbles are needed in e_1. To reach the $m-1$ edges which are incident with the central vertex exactly one edge has to be crossed. Here in this case $2^2(m-1)$ pebbles are needed in e_1. Altogether, for all the cases considered above $6+2(2^2)+2^3(2m-5)+2^2(m-1)$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{2,m}(m \geq 3)$

$$= 6+2(2^2)+2^3(2m-5)+2^2(m-1)+1$$

$$= 20m-29.$$

Theorem 2.9. *For* $J_{3,m}(m \geq 3)$, $CP_E(J_{3,m}) = 72m-139$.

Proof. $J_{3,m}(m \geq 3)$ has $3m+1$ vertices and $4m$ edges. For $J_{3,m}(m \geq 3)$ the edges of the cycle C_{3m} for which both the endpoints are not adjacent with the central vertex are the key edges. Choose any one of the key edges. Let it be e_1. Now e_1 is exactly adjacent with 2 edges which are from the cycle C_{3m}. To place one pebble on each of these 2 edges, $2 * 2 = 4$ pebbles are needed in e_1 because of adjacency. Another two edges from the cycle C_{3m} can be reached by crossing exactly one edge and again another two edges can be reached by crossing exactly two edges on finding the shortest path. As a whole to reach these four edges $2(2^2)+2(2^3)$ pebbles are needed in e_1.

We know that m edges are incident with the central vertex. Out of these m edges, 2 edges can be reached by crossing exactly one edge and the remaining $m-2$ edges can be reached by crossing exactly two edges. Also, with each of these $m-2$ edges, a pair of edges from the cycle C_{3m} are adjacent. In order to reach these $2(m-2)$ edges from cycle C_{3m}, three edges have to be crossed on finding the shortest path. Therefore $2(2^2)+2^3(m-1)$ $+2(m-2)(2^4)$ pebbles are needed in e_1 to reach the edges considered above. All the edges which are incident with the central vertex are dealt with. Only few more edges from the cycle C_{3m} are left. In C_{3m}, totally there are $3m$ edges. Among these $3m$ edges, $7+2(m-2)$ edges are already dealt with. Now, $[3m-7-2(m-2)]$ edges are there to deal with. On finding the shortest path four edges has to be crossed to reach these edges. Hence $[3m-7-2(m-2)]2^5$ pebbles are needed in e_1. All the edges except the chosen key edge e_1 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on e_1.

Hence the cover edge pebbling number of $J_{3,m}(m \geq 3)$ is

$$= 4+2(2^2)+2(2^3)+2(2^2)+2^3(m-2)+2(m-2)(2^4)$$

$$+[3m-7-2(m-2)]2^5+1$$

$$= 72m-139.$$

Theorem 2.10. *For* $J_{4,m}(m \geq 3)$, $CP_E(J_{4,m}) = 104m-167$.

Proof. $J_{4,m}(m \geq 3)$ has $4m+1$ vertices and $5m$ edges. For $J_{4,m}(m \geq 3)$ the edges of the cycle C_{4m} for which both the endpoints are not adjacent with the central vertex are the key edges. Choose any one of the key edges. Let it be e_1. Among the m edges which are incident with the central vertex, one edge can be reached by crossing exactly one edge and the remaining $m-1$ edges can be reached by crossing exactly two edges on finding the shortest path. Therefore, the minimum number of pebbles needed to reach the edges which are incident with the central vertex is $2^2+(m-1)2^3$. Now let us discuss the edges on the cycle C_{4m}. Two edges from the cycle are adjacent with the key edge. Another two edges can be reached by crossing exactly one edge. One pair of edge can be reached by crossing 2 edges and another one pair can be reached by crossing 3 edges. We know that m edges are incident with the central vertex. With each of these m edges two edges from the cycle C_{4m} are adjacent. Among these $2m$ edges from the cycle C_{4m} four edges are already dealt with. The remaining $2m-4$ edges can be reached by crossing 3 edges. Now $9+2m-4$ edges from the cycle C_{4m} are dealt with. Now, remaining $4m-(9+2m-4)$ can be reached by crossing exactly 4 edges on finding the shortest path. Therefore, minimum number of pebbles needed in e_1 to reach the edges of the cycle is $(2*2)+(2^2*2)+(2^3*2)+(2^4*2)$ $+(2m-4)2^4+(4m-(9+2(m-2)))2^5$. After covering all the edges one more pebble has to be placed on e_1. Hence the cover edge pebbling number of $J_{4,m}(m \geq 3)$ is

$$2^2+(m-1)2^3+(2*2)+(2^2*2)+(2^3*2)+(2^4*2)+(2m-4)2^4$$
$$+(4m-(9+2(m-2)))2^5+1$$
$$=104m-167.$$

Theorem 2.11. *For* $J_{5,m}(m \geq 3)$, $CP_E(J_{5,m})=336m-659$.

Proof. $J_{5,m}(m \geq 3)$ has $5m+1$ vertices and $6m$ edges. Let us label the vertices of the cycle as $v_1, v_2, v_3, \ldots, v_{5m}$ in a clockwise manner such that $\deg(v_1)=3$. Now $v_{3+5i}v_{4+5i} \in E(C_{5m})$, the edge set of C_{5m} where $i=0, 1, 2, \ldots, m-1$ are the key edges. Totally there are m key edges. Without loss of generality, let us choose v_3v_4 to be the key edge. Two edges from the cycle C_{5m} are adjacent with v_3v_4 to reach those adjacent edges we need $2*2$ pebbles on v_3v_4. There are five pairs of edges on C_{5m} from which one pair can be reached by crossing one edge, another pair by crossing two edges, another one by three, next pair by four and another one pair by crossing five edges on finding the shortest path. Therefore, to reach these five pairs of edges we need $(2^2*2)+(2^3*2)+(2^4*2)+(2^5*2)+(2^6*2)$ pebbles on v_3v_4. Among m edges which are incident with the central vertex, two edges can be reached by crossing two edges. Therefore, we need $2(2^3)$ pebbles on v_3v_4. And the remaining $m-2$ edges can be reached by crossing three edges and for that $(m-2)(2^4)$ pebbles are needed. Also, with each of these $m-2$ edges which are incident with the central vertex two edges from the cycle C_{5m} are adjacent. To reach these $2(m-2)$ edges of the cycle C_{5m} four edges have to be crossed on finding the shortest path. Therefore, we need $2(m-2)(2^5)$ pebbles on v_3v_4. The edges in the set $\{v_{3+5i}v_{4+5i} / 2 \leq i \leq m-2\}$ can be reached by crossing six edges from v_3v_4. Hence $2^7(m-3)$ are needed. Exactly two edges are adjacent to each of the edges of the set $\{v_{3+5i}v_{4+5i} / 2 \leq i \leq m-2\}$. To reach these adjacent edges we need $(2(m-3))2^6$ pebbles. All the edges except the chosen key edge v_3v_4 have been dealt with. After covering all the edges with one pebble, one more pebble has to be placed on v_3v_4. Hence the cover edge pebbling number for $J_{5,m}(m \geq 3)$ is $(2*2)+(2^2*2)+(2^3*2)+(2^4*2)+(2^5*2)+(2^6*2)+2(2^3)$ $+(m-2)(2^4)+2(m-2)(2^5)+2^7(m-3)+(2(m-3))2^6+1 = 336m-659$.

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