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Aim & Scope

The Advances and Applications in Mathematical Sciences (ISSN 0974-6803) is a monthly journal. The AAMS's coverage extends across the whole of mathematical sciences and their applications in various disciplines, encompassing Pure and Applied Mathematics, Theoretical and Applied Statistics, Computer Science and Applications as well as new emerging applied areas. It publishes original research papers, review and survey articles in all areas of mathematical sciences and their applications within and outside the boundary.

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Contents

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κ - Extensibility and Weakly κ - Extensibility for Some Special Types of Graphs

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ABSTRACT

In this paper we discuss the k-extensibility and weakly k-extensibility for some special types of graphs like Bull graph, Butterfly graph, Durer graph, Friendship graph, Crown graph and Banana tree. Also some theorems, lemmas and corollaries are discussed.

2010 Mathematics Subject Classification: 05C69.

Keywords: k-extensibility in graphs, weakly k-extensibility in graph.

1. INTRODUCTION

A simple graph is a finite non-empty set of object called vertices together with a set of unordered pairs of distinct vertices called edges. A graph G with vertex set V (G) and edge set E (G) is denoted by G = (V , E). The edge e = {u , v} and is denoted by e = uv and it is said that e joins the vertices u and v; and u and v are called adjacent vertices; u and v said to be incident with e. If two vertices are not joined by an edge, then they are said to be nonadjacent.

If two distinct edges are incident with a common vertex, then they are said to be adjacent to each other. A set of vertices in a graph G is independent if no two vertices are adjacent and independent number is denoted by $\beta_0(G)$. An independent set is said to be maximal, if it is not a subset of any larger independent set. Let G = (V , E) be a simple graph. Let k be a positive integer. G is said to be k-extendable if every independent set of cardinality k in G is contained in a maximum independent set of G. Let G be a graph. Let k be a positive integer, $1 \leq k \leq V(G)$. G is said to be weakly k-extendable if every non-maximal independent set of cardinality k of G is contained in a maximum independent set of G.

2. BULL GRAPH

Definition 2.1. The bull graph is a planar undirected graph with 5 vertices and 5 edges. In the form of a triangle with two distinct pendent edges

Example 2.2. Let G:

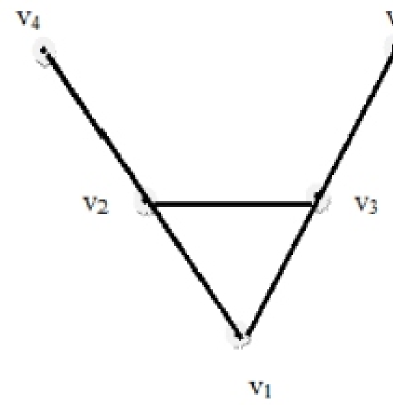

Figure 1

The β_0- set is $\{v_1, v_4, v_5\}$. Here $\beta_0(G) = 3$. $\{v_2\}, \{v_2, v_5\}$ are all independent set of cardinality 1 and 2. Which is not contained in the above β_0- sets of G. Therefore bull graph is not 1-extendable and not 2 extendable. It is only $\beta_0(G)$- extendable graph. It is also not weakly 1-extendable, not weakly 2-extendable for all $k, 1 \leq k \leq (\beta_0(G) - 1)$.

3. BUTTERFLY GRAPH

Definition 3.1. The butterfly graph also called the bowtie graph and the hourglass graph is a planer undirected graph with 5 vertices and 6 edges.

Example 3.2. Let G:

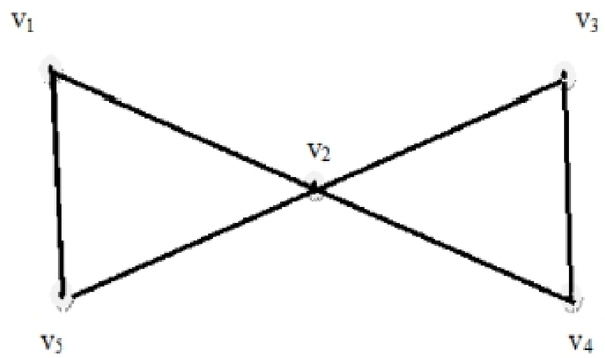

Figure 2

The β_0- sets are $\{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}$. Here $\beta_0(G) = 2$. Since $\{v_2\}$ independent set of cardinality 1 and non-maximal independent set of cardinality 1 which is not contained in any of the above β_0- sets of G. Hence G is not 1 extendable as well as not weakly 1-extendable.

4. DURER GRAPH

Definition 4.1. The Durer graph is an undirected graph with 12 vertices and 18 edges.

Example 4.2. Let G:

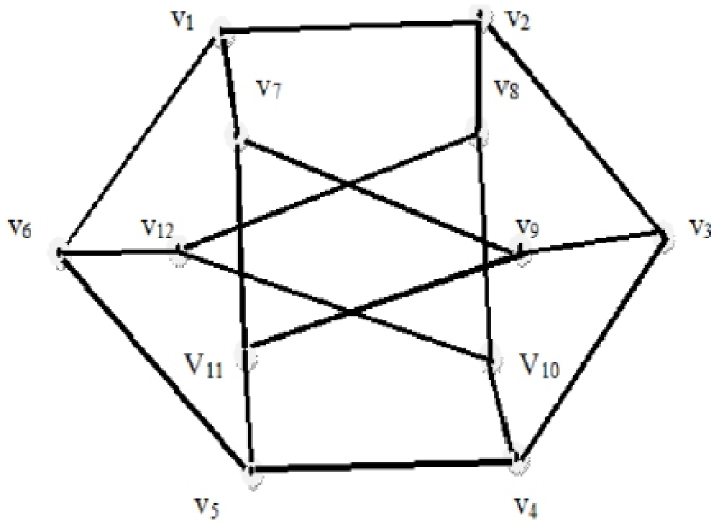

Figure 3

Here $\beta_0(G) = 4$. The β_0- sets are $\{v_1, v_3, v_5, v_8\}, \{v_1, v_3, v_5, v_{10}\}, \{v_1, v_3, v_5, v_{12}\}$ $\{v_2, v_4, v_6, v_7\}, \{v_2, v_4, v_6, v_9\}, \{v_2, v_4, v_6, v_{11}\}, \{v_3, v_5, v_7, v_8\}, \{v_4, v_6, v_7, v_8\}$, $\{v_4, v_6, v_8, v_9\}, \{v_1, v_5, v_8, v_9\}, \{v_1, v_5, v_9, v_{10}\}, \{v_2, v_6, v_9, v_{10}\}, \{v_2, v_6, v_{10}, v_{11}\}$, $\{v_1, v_3, v_{10}, v_{11}\}, \{v_1, v_3, v_{11}, v_{12}\}, \{v_2, v_4, v_{11}, v_{12}\}$. Any independent set of cardinality k is contained in any one of the above β_0- sets of G. Therefore G is k-extendable for all k, $1 \leq k \leq \beta_0(G)$ and it is also weakly k-extendable for all k, $1 \leq k \leq (\beta_0(G) - 1)$.

5. FRIENDSHIP GRAPH

Definition 5.1. The friendship graph (or Dutch windmill graph or n-fan) F_n is a planer undirected graph with 2_{n+1} vertices and 3_n edges.

Lemma 5.2. *Let $G = F_n$ be Friendship, F_n is k-extendable graph for all k, $2 \leq k \leq \beta_0(G)$ except at $k = 1$.*

Proof. Let $G = F_n$ be a Friendship graph. Let D be an independent set of G cardinality k, $D = \{u_{i1}, u_{i2}, u_{i3}, \ldots, u_{ir}\} \cup \{v_{j1}, v_{j2}, v_{j3}, \ldots, v_{js}\}$, where $r + s = k$ and $i_e \neq j_m$ for all l, m. The maximum independent set is $D \cup \{u_{si} : i \neq i_a$ and $i \neq j_a$ for all $a, b, 1 \leq a \leq r \quad 1 \leq b \leq s\}$ which contains any independent set of cardinality k, for all $k,\ 2 \le k \le \beta_0(G)$. Also, G contains only one-full degree vertex, which is not contained in D. Therefore G is not one-extendable. Hence G is k-extendable graph for all $k,\ 2 \le k \le \beta_0(G)$, except at $k = 1$.

6. CROWN GRAPH

Definition 6.1. The n-Crown graph an integer n ≥ 3 is the graph with vertex set $\{x_0, x_1, x_2, \ldots, x_{n-1}, y_0, y_1, y_2, \ldots, y_{n-1}\}$ and edge set $\{(x_i, y_i) : 0 \le i, j \le n-1, i \ne j\}$.

Theorem 6.2. Let G be a crown graph, (n ≥ 3) G is k-extendable graph for all k≤ 1 k≤ β0 (G) , except at k = 2 .

Proof. Let G be a crown graph with vertices (n ≥ 3).

The vertex set of G be $V(G) = \{x_0, x_1, x_2, \ldots, x_{n-1}, y_0, y_1, y_2, \ldots, y_{n-1}\}$ and the edge set of G be $E(G) = \{(x_i, y_i) : 0 \le i, j \le n-1, i \ne j\}$. The maximum independent sets of G are $\{x_0, x_1, x_2, \ldots, x_{n-1}\}, \{y_0, y_1, y_2, \ldots, y_{n-1}\}$. Here $\beta_0(G) = n$. By the definition of Crown graph, there is no edge between (x_i, y_j) where $i = j$. That is $\{x_i, y_i\}$, is the independent set of cardinality two which is not contained in the above maximum independent set of G. Therefore G is not 2-extendable. From the construction of the maximum independent sets of G, it is clear that G is k-extendable for all $k,\ 1 \le k \le \beta_0(G)$ except at $k = 2$. Hence the proof.

Corollary 6.3. In crown graph (n ≥ 3), we have n number of maximal independent sets of cardinality two. So we have non-maximal independent set of cardinality one, that clearly contains in any one of the maximum independent set of G. Therefore G is weakly 1-extendable.

7. BANANA TREE

Definition 7.1. An (B_{nk}) , banana tree graph obtained by connecting one leaf of each of n copies of an k-star graph with a single root vertex that is distinct from all the stars.

Theorem 7.2. *Let* $G = B_{n,k}$, *if* $\beta_0(n, k) = nk - n$, *then* G *is not extendable for all* $k,\ 1 \le k \le (nk - (2n - 1))$ *and it is not weakly extendable for all* $k,\ 1 \le k \le (nk - (2n))$.

Proof. Let $G = B_{n,k}$. To prove this theorem by using induction on n, Let $n = 2$, $G = B_{2,k}$. The vertex set of $B_{2,k}$ be $V_1 = \{v_1, v_2, \ldots, v_{2k+1}\}$. The graph of $G = B_{2,k}$ is given below,

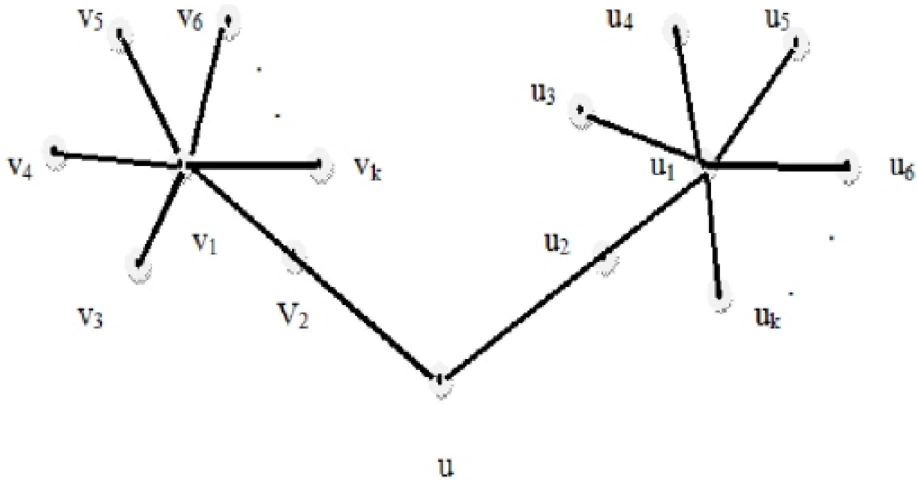

Figure (4)

From the above graph, we seen that the maximum independent set is, $\{v_2, v_3, \ldots, v_k, u_2, u_3, \ldots, u_k\}$, $\beta_0(B_{2,k}) = 2k - 2$ $\cdot$ $\{u\}$, $\{u, v_3\}$, $\{u, v_3, v_4\}$, $\{u, v_3, v_4, v_5\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k\}$, $\{u, v_3, v_4, v_5, \ldots, v_k, u_3\}$, $\{u, v_3, v_4, v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k\}$ are all independent set of cardinality k, for all $k, 1 \leq k \leq (2k - 3)$, which is not contained in the above β_0- sets of $B_{2,k}$. Therefore $B_{2,k}$ is not k-extendable for all $k, 1 \leq k \leq (\beta_0(B_{2,k}) - 1)$.

$\{u, v_3, v_4, v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_{k-1}\}$ are all non-maximal independent set of $B_{2,k}$, which is not contained in the above β_0- sets of $B_{2,k}$. Hence $B_{2,k}$ is not weakly k-extendable for all $k, 1 \leq k \leq (\beta_0(B_{2,k}) - 2)$. Suppose $n = 3$, $G = B_{n,k}$. The vertex set of $B_{3,k}$ be $V_2 = \{v_1, v_2, \ldots, v_{3k+1}\}$. The graph of $G = B_{3,k}$ is given below,

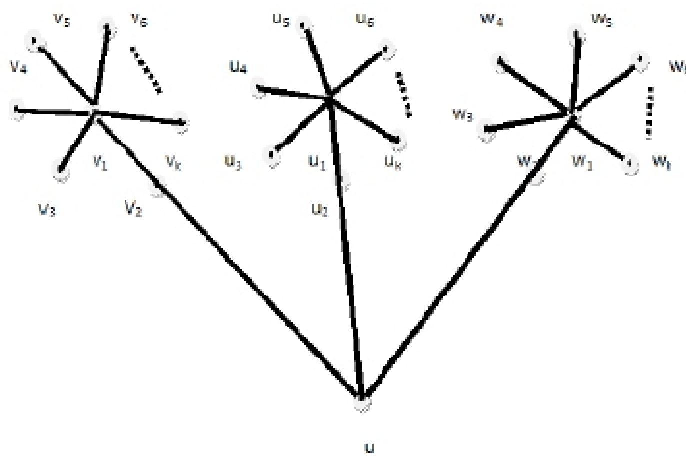

Figure 5

From the above graph, we seen that the maximum independent set is, $\{v_2, v_3, \ldots, v_k, u_2, u_3, \ldots, u_k, w_2, w_3, \ldots, w_k\}$, $\beta_0(B_{3,k}) = 3k - 3$.

$\{u\}, \{u, v_3\}, \{u, v_3, v_4\}, \{u, v_3, v_4, v_5\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k, w_3, w_4, \ldots, w_k\}$ are all independent set of cardinality k, for all $1 \le k \le (3k - 5)$, which is not contained in the above β_0-sets of $B_{3,k}$. Therefore $B_{3,k}$ is not k-extendable for all $k, 1 \le k \le (\beta_0(B_{3,k}) - 2)$. $\{u, v_3, v_4, v_5, \ldots, v_k, u_3, u_4\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k\}, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k, w_3\}, \ldots, \{u, v_3, v_4, v_5, \ldots, v_k, u_3, \ldots, u_k, w_3, w_4, \ldots, w_{k-1}\}$ are all non-maximal independent set of $B_{3,k}$, which is not contained in the above β_0-sets of $B_{3,k}$. Hence $B_{3,k}$ is not weakly k-extendable for all $k, 1 \le k \le (\beta_0(B_{3,k}) - 3)$. Suppose the result is true for n, $B_{n,k}$ is not k-extendable for all $k, 1 \le k \le nk - (2n - 1)$. Also $B_{n,k}$ is not weakly k-extendable for all $k, 1 \le k \le nk - (2n)$. Since the result is true for n, that means β_0-set of $B_{n,k}$ contains $(nk - n)$-vertices. In $B_{n+1,k}$, the graph $B_{n+1,k}$ is partition into two graphs one is $B_{n,k}$ and another one is a subgraph which is nothing but $K_{1,n-1}$. Let $B_{n+1,k}$. The vertex set of $B_{n+1,k}$ be $V = \{v_1, v_2, \ldots, v_{(nk+1)+k}\}$. The graph of $G = B_{n+1,k}$ is given below,

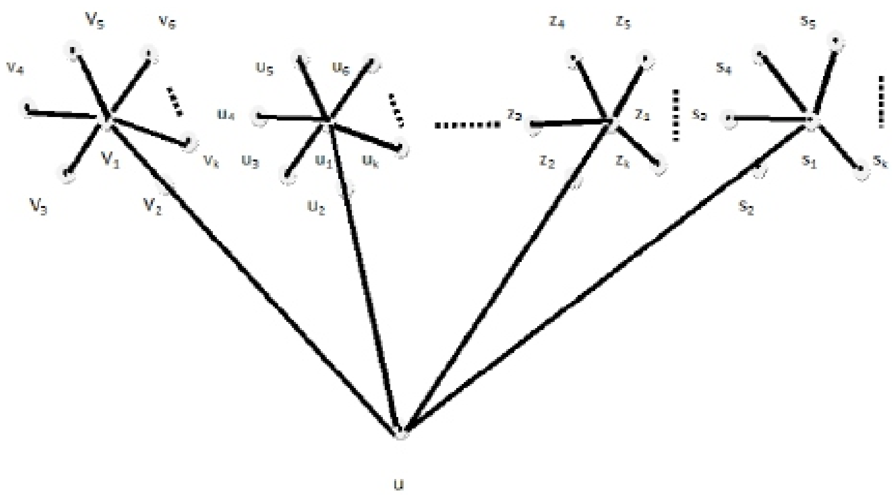

Figure 6

From the above graph, we seen that the maximum independent set is, $\{v_2, v_3, \ldots, v_k, u_2, u_3, \ldots, u_k, w_2, w_3, \ldots, w_k, \ldots, z_1, z_2, \ldots, z_k, s_1, s_2, \ldots, s_k\}$.

Therefore $\beta_0(B_{n+1,k}) = nk - n + (k-1) \cdot \{u\}, \{u, v_3\}, \{u, v_3, v_4\}, \{u, v_3, v_4, \ldots, v_k\}, \ldots, \{u, v_3, \ldots, v_k, u_3, \ldots, u_k, \ldots, z_1, \ldots, z_k, S_3, \ldots, S_k\}$ are all independent set of $B_{n+1,k}$, which is not contained in the above β_0- sets of $B_{n+1,k}$. Therefore $B_{n+1,k}$ is not k-extendable for all $k, 1 \leq k \leq 1 + (nk - 2n) + (k-2)\{u\}, \{u, v_3\}, \{u, v_3, v_4\}, \{u, v_3, v_4, \ldots, v_k\}, \ldots, \{u, v_3, \ldots, v_k, u_3, \ldots, u_k, \ldots, z_1, \ldots, z_k, s_3, \ldots, s_{k-1}\}$ are all non-maximal independent set of $B_{n+1,k}$, which is not contained in the above β_0- sets of $B_{n+1,k}$. Therefore $B_{n+1,k}$ is not weakly k-extendable for all $k, 1 \leq k \leq 1 + (nk - 2n) + (k-3)$. Hence the proof.

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Indegree Prime Labeling of Some Special Directed Graphs

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ABSTRACT

Let $D(p, q)$ be a digraph. A function $f: V \rightarrow \{1, 2, 3, \ldots\ldots, p+q\}$ is said to be an in degree prime labeling of D if at each $v \in V(D)$, $gcd[f(u), f(v)] = 1\ \forall\ uv \in E(D)$. In this paper, we investigate in degree prime labeling of some special directed graphs. We prove that the directed graphs such as Instar IK 1, n Inwheel IW n, Upcomb $U_p(P_n \odot K_1)$ and Incrown $I(C_n \odot K_1)$graphs admit Indegree prime labeling.

2010 Mathematics Subject Classification: 05C78.

Keywords: Indegree prime labeling, Instar, Inwheel, Upcomb, Incrown.

1. INTRODUCTION

All graphs in this paper are finite and directed. A finite graph is a graph G (V , E) in which the vertex set and the edge set are finite sets. A directed graph D (V , A) is an ordered pair of set of vertices (nodes) V (D) and the set of arcs (directed edges) A (D). Any arc (x , y) is a directed edge from x to y. The pair (x , y) is an ordered pair in which the first component (initial vertex) x is called the tail of the arc and the second component (terminal vertex) y is called the head of the arc. A digraph D is a Strict Digraph if it has no loops and no two arcs with the same end vertices. Throughout this paper we consider only strict digraphs. A digraph D with p vertices and q arcs is denoted by D (p , q). The indegree $d(v)^-$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The outdegree $d(v)^+$ of v is the number of arcs having v as its initial vertex [6]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The notion of a prime labeling originated with Entringer and was introduced in a paper by Hameeda Begum N. Tout, R. Rubina and M. I. Jawahar Nisha Dabboucy and Howalla. A graph with vertex set V is said to have a prime labeling of its vertices if its vertices are labeled with distinct integers {1, 2, 3,... |V|} such that for each edge xy, the labels assigned to x and y are relatively prime [4].

2. PRELIMINARIES

For the following preliminary definitions we refer [1] and [5]

Definition 2.1. Let $D(p, q)$ be a digraph. A function $f: V \rightarrow \{1, 2, 3, \ldots, p+q\}$ is said to be an indegree prime labeling of D if at each $v \in V(D)$, gcd $[f(u), f(v)] = 1 \forall \overrightarrow{uv} \in E(D)$.

Definition 2.2. A star graph $K_{1,n}$ in which all the edges are directed towards the root vertex is called an Instar and is denoted as $IK_{1,n}$.

Definition 2.3. A wheel graph W_n in which the edges of the outer cycle are directed clockwise or anticlockwise and the spoke edges are directed towards the central vertex (Hub vertex) is called an In wheel and is denoted by IW_n or $I(C_n \odot K_{1,n})$.

Definition 2.4. A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendent edges are oriented away from the end vertices is called an Upcomb and is denoted by $U_p(P_n \odot K_1)$.

Definition 2.5. A crown graph $C_n \odot K_1$ in which the edges of cycle are directed clockwise or anticlockwise and the pendent edges are directed towards the cycle is called an Incrown and is denoted by $I(C_n \odot K_1)$.

For the following remarks we refer [2]

Remark 2.6. 1 is relatively prime with all natural numbers.

Remark 2.7. 2 is relatively prime with all odd natural numbers.

Remark 2.8. Any two consecutive natural numbers is relatively prime.

Remark 2.9. Any two consecutive odd natural numbers is relatively prime.

3. MAIN RESULTS

Theorem 3.1. Instar (3) IK 1, n ≥ n admits Indegree Prime Labeling.

Proof. Let D be an Instar $I\!K_{1,n}$ with vertex set $V(D) = \{v, v_1, v_2, v_3, \dots, v_n\}$, where v is the root vertex and the arc set be $E(D) = \{\overrightarrow{v_1 v}, \overrightarrow{v_2 v}, \overrightarrow{v_3 v}, \dots, \overrightarrow{v_n v}\}$.

Then the Instar $I\!K_{1,n}$ is as in figure 3.1(a)

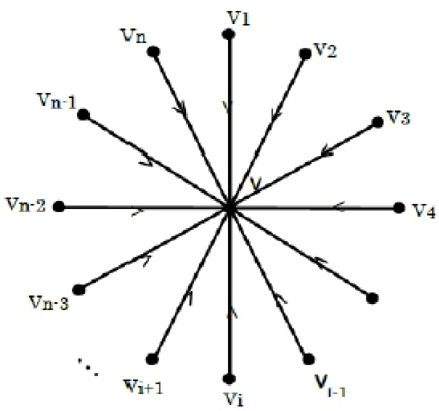

Figure 3.1(a). Instar IK1,n

Here $p = n+1$, $q = n$. So, $p + q = 2n + 1$. We define a function $f : V \to \{1, 2, 3, \dots, 2n+1\}$ by $f(v) = 1$ and $f(v_i) = i + 1$, $1 \le i \le n$. Indegree of v is n, $d^-(v) = n$. Indegree of v_i is zero, $d^-(v_i) = 0$, $\forall 1 \le i \le n$.

Then by remark 2.6, at $v \in V(D)$, gcd $[f(v_i), f(v)] = 1$, $\forall \overrightarrow{v_i v} \in E(D)$, since gcd $[i+1, 1] = 1 \; \forall \; 1 \le i \le n$.

Therefore, In view of the above labeling pattern, it is evident that the Instar $IK_{1,n}$ admits an In degree Prime Labeling.

Illustration 3.1.1. Indegree Prime Labeling of $IK_{1,12}$ is shown in Figure 3.1(b)

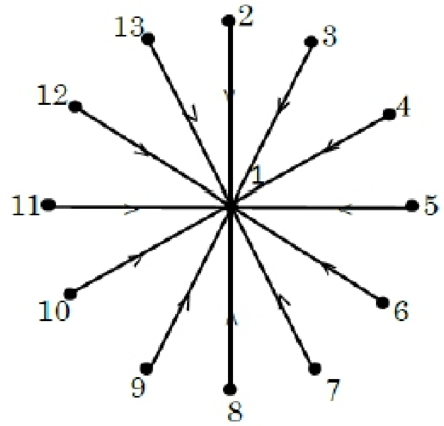

Figure 3.1(b). Instar $IK_{1,12}^-$

Theorem 3.2. *In wheel IW_n $(n \ge 3)$ admits Indegree Prime Labeling.*

Proof. Let D be an Inwheel IW_n with the vertex set $V(D) = \{v, v_1, v_2, v_3, \ldots, v_n\}$, where v is the central vertex and the arc set be $E(D) = \{\overrightarrow{v_1 v}, \overrightarrow{v_2 v}, \overrightarrow{v_3 v}, \ldots, \overrightarrow{v_{n-1} v}, \overrightarrow{v_n v}\} \cup \{\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_4}, \ldots, \overrightarrow{v_{n-1} v_n}, \overrightarrow{v_n v_1}\}$.

In general, $E(D) = \{\overrightarrow{v_i v} / 1 \le i \le n\} \cup \{\overrightarrow{v_i v_{i+1}} / 1 \le i \le n-1, \overrightarrow{v_n v_1}\}$.

Then the In wheel IW_n is as in figure 3.2(a)

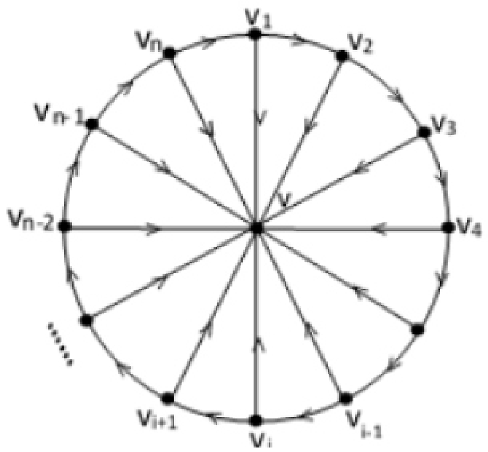

Figure 3.2(a). In wheel IW_n^-

Here $p = n + 1,\ q = 2n$. So, $p + q = 3n + 1$. We define a function $f : V \to \{1, 2, 3, \ldots, 3n + 1\}$ by $f(v) = 2$ and $f(v_i) = 2i - 1, 1 \le i \le n$.

Indegree of v is n, $d^-(v) = n$.

Indegree of v_i is 1, $d^-(v_i) = 1, \forall 1 \le i \le n$.

Then at $v \in V(D)$, gcd $[f(v_i), f(v)] =$ gcd $[2i - 1, 2] = 1, \forall\ \overrightarrow{v_i v} \in E(D)$, where $1 \le i \le n$. (by remark 2.7)

Also at $v_i \in V(D)$ for $2 \le i \le n$; gcd $[f(v_{i-1}), f(v_i)] =$ gcd $[2i - 3, 2i - 1] = 1$, $\forall\ \overrightarrow{v_{i-1} v_1} \in E(D)$, where $2 \le i \le n$ (by remark 2.9).

Also at $v_1 \in V(D)$, gcd $[f(v_n), f(v_1)] =$ gcd $[2n - 1, 1] = 1$, $\overrightarrow{v_n v_1} \in E(D)$, (by remark 2.6).

Therefore, in view of the above labeling pattern, it is evident that the Inwheel IW_n admits an Indegree Prime Labeling (irrespective of n being odd or even).

Illustration 3.2.1. Indegree Prime Labeling of IW_{13} is shown in Figure 3.2(b)

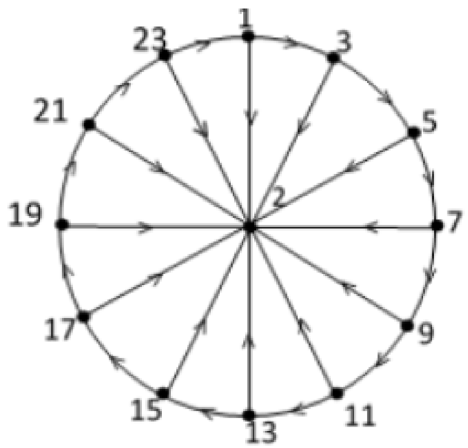

Figure 3.2(b). Instar IK_{13}^-

Theorem 3.3. *Upcomb $U_p(P_n \odot K_1)$ admits Indegree Prime Labeling.*

Proof. Let D be an Upcomb $U_p(P_n \odot K_1)$ with the vertex set $V(D) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n\}$, i.e., $V(D) = \{v_i, u_i / 1 \le i \le n\}$, where v_i represents the vertices of the path and u_i represents the pendent vertices corresponding to each v_i respectively and the arc set

$E(D) = \{\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_4}, \ldots, \overrightarrow{v_{n-1} v_n}\} \cup \{\overrightarrow{u_1 v_1}, \overrightarrow{u_2 v_2}, \overrightarrow{u_3 v_3}, \ldots, \overrightarrow{u_{n-1} v_{n-1}}, \overrightarrow{u_n v_n}\}$

i.e., $E(D) = \{\overrightarrow{v_i v_{i+1}} / 1 \le i \le n-1\} \cup \{\overrightarrow{u_i v_i} / 1 \le i \le n\}$.

Then the Upcomb $U_p(P_n \odot K_1)$ is as in figure 3.3(a)

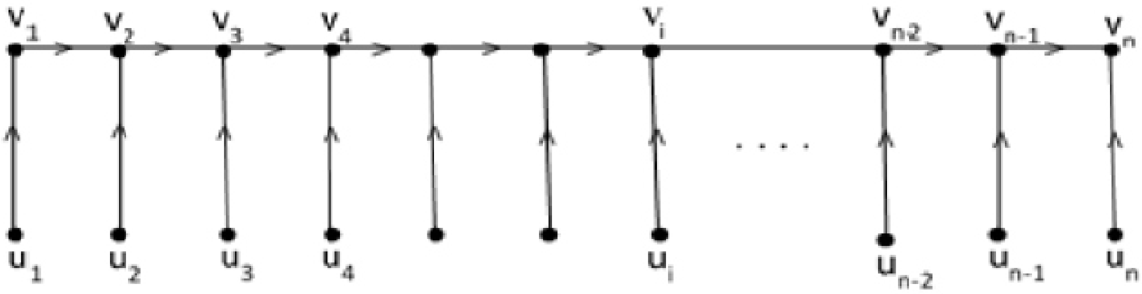

Figure 3.3(a). Upcomp $U_p(P_n \odot K_1)$.

Here $p = 2n$, $q = 2n - 1$. So, $p + q = 4n - 1$. We define a function $f : V \to \{1, 2, 3, \ldots, 4n - 1\}$ by $f(v_i) = 2i - 1, 1 \le i \le n$ and $f(u_i) = 2i$, $1 \le i \le n$. Indegree of u_i is 0, $d^-(u_i) = 0$. Indegree of v_1 is 1, $d^-(v_1) = 1$, since $\overrightarrow{u_1 v_1}$ is the only arc giving indegree to v_1. Indegree of v_i is 2, $d^-(v_i) = 2$; $\forall\ 2 \le i \le n$, since $\overrightarrow{v_{i-1} v_i}$ and $\overrightarrow{u_i v_i}$ for $2 \le i \le n$, are the only two arcs giving indegree to v_i.

Then at $v_1 \in V(D)$, gcd $[f(u_1), f(v_1)] =$ gcd $[2, 1] = 1$, $\overrightarrow{u_1 v_1} \in E(D)$ (by remark 2.6).

Also at $v_i \in V(D)$ for $2 \le i \le n$, gcd $[f(v_{i-1}), f(v_i)] =$ gcd $[2i - 3, 2i - 1] = 1$, $\forall\ \overrightarrow{v_{i-1} v_i} \in E(D)$, where $2 \le i \le n$. (by remark 2.9) and gcd $[f(u_i), f(v_i)] =$ gcd $[2i, 2i - 1] = 1$, $\forall\ \overrightarrow{u_i v_i} \in E(D)$, where $2 \le i \le n$ (by remark 2.8).

Therefore, in view of the above labeling pattern, it is evident that the Upcomb $U_p(P_n \odot K_1)$ admits an Indegree Prime Labeling.

Illustration 3.3.1. Indegree Prime Labeling of $U_p(P_{11} \odot K_1)$ is shown in Figure 3.3(b).

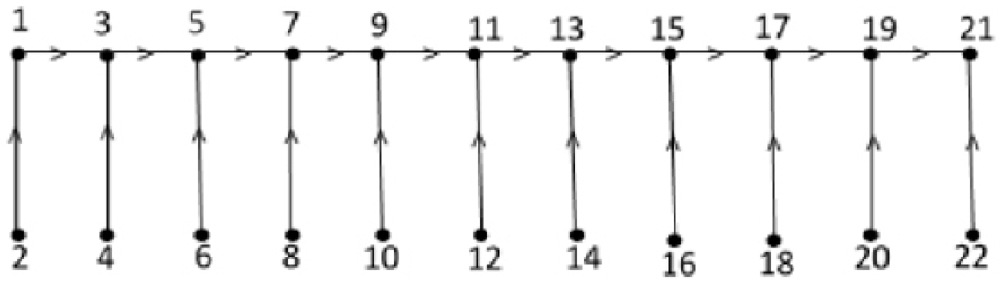

Figure 3.3(b). Upcomb $U_p(P_{11} \odot K_1)$.

Theorem 3.4. *Incrown* $I(C_n \odot K_1)$ *admits Indegree Prime Labeling.*

Proof. Let D be an Incrown $I(C_n \odot K_1)$ with the vertex set $V(D) = \{v_1, v_2, v_3, \dots, v_n, u_1, u_2, u_3, u_n\}$, i.e., $V(D) = \{v_i, u_i / 1 \le i \le n\}$, where v_i represents the vertices of the cycle and u_i represents the pendent vertices corresponding to each v_i respectively and the arc set

$$E(D) = \{\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_4}, \dots, \overrightarrow{v_{n-1} v_n}, \overrightarrow{v_n v_1}\} \cup \{\overrightarrow{u_1 v_1}, \overrightarrow{u_2 v_2}, \overrightarrow{u_3 v_3}, \dots, \overrightarrow{u_{n-1} v_{n-1}}, \overrightarrow{u_n v_n}\}.$$

i.e., $E(D) = \{\overrightarrow{v_i v_{i+1}} / 1 \le i \le n-1, \overrightarrow{v_n v_1}\} \cup \{\overrightarrow{u_i v_i} / 1 \le i \le n\}$.

Then the Incrown $I(C_n \odot K_1)$ is as in figure 3.4(a)

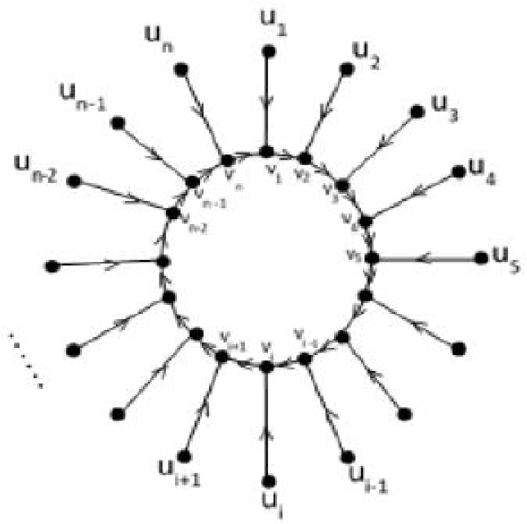

Figure 3.4(a). Incrown $I(C_n \odot K_1)$.

Here $p = 2n$, $q = 2n$. So, $p + q = 4n$. We define a function $f : V \to \{1, 2, 3, \dots, 4n\}$ by $f(v_i) = 2i - 1$, $1 \le i \le n$ and $f(u_i) = 2i$, $1 \le i \le n$. Indegree of u_i is 0, $d^-(u_i) = 0$. Indegree of v_i is 2, $d^-(v_i) = 2$, $\forall\, 1 \le i \le n$.

Since $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{u_iv_i}$ are the only two arcs giving indegree to v_i, for $2 \le i \le n$ and $\overrightarrow{v_nv_1}$ and $\overrightarrow{u_1v_1}$ are the two arcs giving indegree to v_1.

Then at $v_1 \in V(D)$, gcd $[f(v_n), f(v_1)]$ = gcd $[2n - 1, 1] = 1$, $\forall \overrightarrow{v_nv_1} \in E(D)$ (by remark 2.6).

Also at $v_i \in V(D)$, gcd $[f(u_i), f(v_i)]$ = gcd $[2i, 2i - 1] = 1$, $\forall \overrightarrow{u_iv_i} \in E(D)$, where $2 \le i \le n$. (by remark 2.8) and at $v_i \in V(D)$, gcd $[f(v_{i-1}), f(v_i)]$ = gcd $[2i - 3, 2i - 1] = 1$, $\forall \overrightarrow{v_{i-1}v_i} \in E(D)$, (by remark 2.9).

Therefore, In view of the above labeling pattern, it is evident that the Incrown $I(C_n \odot K_1)$ admits an Indegree Prime Labeling.

Illustration 3.4.1. Indegree Prime Labeling of $I(C_{16} \odot K_1)$ is shown in Figure 3.4(b)

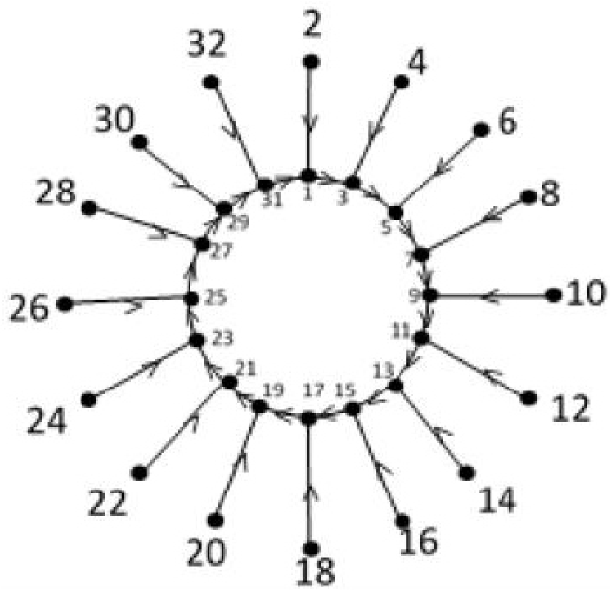

Figure 3.4(b). Incrown $I(C_{16} \odot K_1)$.

4. CONCLUSION

It is very interesting to investigate the directed graphs and its families which admit indegree prime labeling. In this paper we have proved that the Instar $IK_{1,n}$, Inwheel IW_n, Upcomb $U_p(P_n \odot K_1)$ and Incrown $I(C_n \odot K_1)$ graphs admit the Indegree prime labeling. To investigate similar results for other families of directed graphs is an open area of research.

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Information Measure on Bipolar Intuitionistic Fuzzy Soft Sets

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ABSTRACT

The aim of this paper is to introduce operators on bipolar intuitionistic fuzzy soft sets (BIFSS). Properties of operators on BIFSS are established. An information measure on BIFSS is proposed. Making use of the proposed information measure a decision making concept is constructed. Finally, this concept is illustrated by an example.

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Keywords: Bipolar intuitionistic fuzzy soft set, operators on BIFSS, information measure, decision making problem.

1. INTRODUCTION

Zadeh [6] introduced fuzzy sets and Atanassov [2] introduced the IFS. Chiranjibe et.al.,[3] developed the notion of BIFS set. Information measure in fuzzy sets is certainly a measure of fuzziness, while for intuitionistic fuzzy sets information measure measures both the fuzziness and intuitionism. Srinivasan et.al., [5] introduced some operations on intuitionistic fuzzy sets of root type. Anita shanthi et.al., [1] introduced information measure of IVIFSSRT. Mishra [4] developed the intuitionistic fuzzy information measure.

Now, we define operators on BIFSS and study some of its properties. We propose the concept of an information measure of BIFSS. Further decision making concept on information measure of BIFSS is developed.

2. OPERATIONS ON BIFSS

In this section, we define operators on BIFSS. Some properties of these operators are discussed.

Definition 2.1. Let

$$(BF, E) = \{x, ((\mu^{n}_{BF(e)}(x), \mu^{p}_{BF(e)}(x)), ((\nu^{n}_{BF(e)}(x), \nu^{p}_{BF(e)}(x)) : x \in U, e \in E\}$$

be BIFSS. The hesitant degree of BIFSS is denoted by $\pi^{n}_{BF(e)}$ and $\pi^{p}_{BF(e)}$ is defined as

$$\pi^{n}_{BF(e)}(x) = -1 - \mu^{n}_{BF(e)}(x) - \upsilon^{n}_{BF(e)}(x) \text{ and } \pi^{p}_{BF(e)}(x) = 1 - \mu^{p}_{BF(e)}(x) - \upsilon^{p}_{BF(e)}(x)$$

where $\pi^{n}_{BF(e)} \in [-1, 0]$ and $\pi^{p}_{BF(e)} \in [1, 0]$ respectively.

Definition 2.2. Let $\eta \in [0, 1]$ be a fixed number. Given an *BIFSS* (F, E) the operator D_η is defined as

$$D_\eta(BF, E) = \{x, ((\mu^{n}_{BF(e)}(x) + \eta\pi^{n}_{BF(e)}(x)), ((\mu^{p}_{BF(e)}(x) + \eta\pi^{p}_{BF(e)}(x)), ((\nu^{n}_{BF(e)}(x) + (1-\eta)\pi^{n}_{BF(e)}(x)), ((\nu^{p}_{BF(e)}(x) + (1-\eta)\pi^{p}_{BF(e)}(x)) : x \in U, e \in E\}.$$

Definition 2.3. For $\eta, \delta \in [0, 1]$, $\eta + \delta \leq 1$ the operator $F_{\eta, \delta}$ for a *BIFSS* (BF, E) is defined as

$$F_{\eta, \delta}(BF, E) = \{x, ((\mu^{n}_{BF(e)}(x) + \eta\pi^{n}_{BF(e)}(x)), ((\mu^{p}_{BF(e)}(x) + \eta\pi^{p}_{BF(e)}(x)),$$

$$((\nu^{n}_{BF(e)}(x) + \delta\pi^{n}_{BF(e)}(x)), ((\nu^{p}_{BF(e)}(x)$$

$$+ \delta\pi^{p}_{BF(e)}(x)) : x \in U, e \in E\}.$$

Definition 2.4. Let $\eta, \delta \in [0, 1]$. Given a *BIFSS* (BF, E), the operator $G_{\eta, \delta}$ is defined as

$$G_{\eta, \delta}(BF, E) = \{x, ((\eta\mu^{n}_{BF(e)}(x), \eta\mu^{n}_{BF(e)}(x)), ((\delta\nu^{n}_{BF(e)}(x),$$

$$\delta\nu^{p}_{BF(e)}(x)) : x \in U, e \in E\}.$$

Obviously $G_{1,1}(BF, E) = (BF, E)$ and $G_{0,0}(BF, E) = \Phi$.

Definition 2.5. (BF, E) and (BG, E) are two *BIFSS*. The operation @ on (BF, E) and (BG, E) is defined as

$$(BF, E) @ (BG, E) = \left\{x, \left(\frac{\mu^{n}_{BF(e)}(x) + \mu^{n}_{BG(e)}(x)}{2}, \frac{\mu^{p}_{BF(e)}(x) + \mu^{p}_{BG(e)}(x)}{2}\right),\right.$$

$$\left.\left(\frac{\nu^{n}_{BF(e)}(x) + \nu^{n}_{BG(e)}(x)}{2}, \frac{\nu^{p}_{BF(e)}(x) + \nu^{p}_{BG(e)}(x)}{2}\right) x \in U, e \in E\right\}.$$

Theorem 2.6. *Let* (BF, E) *and* (BG, E) *are two BIFSS and* $\eta, \delta \in [0, 1]$, *then the following conditions holds:*

(i) $D_{\eta}((BF, E) @ (BG, E)) = D_{\eta}(BF, E) @ D_{\eta}(BG, E)$

(ii) $F_{\eta, \delta}((BF, E) @ (BG, E)) = F_{\eta, \delta}(BF, E) @ F_{\eta, \delta}(BG, E)$

(ii) $G_{\eta, \delta}((BF, E) @ (BG, E)) = G_{\eta, \delta}(BF, E) @ G_{\eta, \delta}(BG, E)$.

Proof.

(i) $(BF, E) @ (BG, E) = \left\{x, \left(\frac{1}{2}, (\mu^{n}_{BF(e)}(x) + \mu^{n}_{BG(e)}(x), \mu^{p}_{BF(e)}(x) + \mu^{p}_{BG(e)}(x))\right),\right.$

$$\left(\frac{1}{2}, (\nu^n_{BF(e)}(x) + \nu^n_{BG(e)}(x), \nu^p_{BF(e)}(x) + \nu^p_{BG(e)}(x))\right)\Big\}$$

$$D_\eta((BF, E) @ (BG, E)) = \Big\{x, \Big(\frac{1}{2}(\mu^n_{BF(e)}(x) + \eta\pi^n_{BF(e)}(x) + \mu^n_{BG(e)}(x) + \eta\pi^n_{BG(e)}(x),$$

$$\mu^p_{BF(e)}(x) + \eta\pi^p_{BF(e)}(x) + \mu^p_{BG(e)}(x) + \eta\pi^p_{BG(e)}(x))\Big)$$

$$\Big(\frac{1}{2}(\nu^n_{BF(e)}(x) + (1-\eta)\pi^n_{BF(e)}(x) + \nu^n_{BG(e)}(x) + (1-\eta)\pi^n_{BG(e)}(x), \nu^p_{BF(e)}(x)$$

$$+ (1-\eta)\pi^p_{BF(e)}(x) + \nu^p_{BG(e)}(x) + (1-\eta)\pi^p_{BG(e)}(x))\Big)\Big\}$$

$$D_\eta(BF, E) = \{x, (\mu^n_{BF(e)}(x) + \eta\pi^n_{BF(e)}(x) + \mu^p_{BF(e)}(x) + \eta\pi^p_{BF(e)}(x))$$

$$(\nu^p_{BF(e)}(x) + (1-\eta)\pi^p_{BF(e)}(x), \nu^p_{BF(e)}(x) + (1-\eta)\pi^p_{BF(e)}(x))\}$$

$$D_\eta(BG, E) = \{x, (\mu^n_{BG(e)}(x) + \eta\pi^n_{BG(e)}(x), \mu^p_{BG(e)}(x) + \eta\pi^p_{BG(e)}(x)),$$

$$(\nu^p_{BG(e)}(x) + (1-\eta)\pi^n_{BG(e)}(x), \nu^p_{BG(e)}(x) + (1-\eta)\pi^p_{BG(e)}(x))\}$$

$$D_\eta(BF, E) @ D_\eta(BG, E)$$

$$= \{x, \frac{1}{2}(\mu^n_{BF(e)}(x) + \eta\pi^n_{BF(e)}(x) + \mu^n_{BG(e)}(x) + \eta\pi^n_{BG(e)}(x))$$

$$\mu^p_{BF(e)}(x) + \eta\pi^p_{BF(e)}(x) + \mu^p_{BG(e)}(x) + \eta\pi^p_{BG(e)}(x))),$$

$$(\frac{1}{2}(\nu^n_{BF(e)}(x) + (1-\eta)\pi^n_{BF(e)}(x) + \nu^n_{BG(e)}(x) + (1-\eta)\pi^n_{BG(e)}(x)),$$
$$\nu^p_{BF(e)}(x) + (1-\eta)\pi^p_{BF(e)}(x) + \nu^p_{BG(e)}(x) + (1-\eta)\pi^p_{BG(e)}(x)))\}.$$

Therefore $D_\eta((BF, E) @ (BG, E)) = D_\eta(BF, E) @ D_\eta(BG, E)$.

(ii) $F_{\eta,\delta}((BF, E) @ (BG, E)) = \{x, \frac{1}{2}(\mu^n_{BF(e)}(x) + \eta\pi^n_{BF(e)}(x) + \mu^n_{BG(e)}(x)$
$+ \eta\pi^n_{BG(e)}(x)), \mu^p_{BF(e)}(x) + \eta\pi^p_{BF(e)}(x) + \mu^p_{BG(e)}(x) + \eta\pi^p_{BG(e)}(x))),$
$(\frac{1}{2}(\nu^n_{BF(e)}(x) + \delta\pi^n_{BF(e)}(x) + \nu^n_{BG(e)}(x) + \delta\pi^n_{BG(e)}(x)),$

$$\upsilon^{p}_{BF(e)}(x)+\delta\pi^{p}_{BF(e)}(x)+\nu^{p}_{BG(e)}(x)+\delta\pi^{p}_{BG(e)}(x)))\}$$

$$F_{\eta,\delta}(BF, E)=\{x,(\mu^{n}_{BF(e)}(x)+\eta\pi^{n}_{BF(e)}(x)+\mu^{p}_{BF(e)}(x)+\eta\pi^{p}_{BF(e)}(x)),$$

$$(\nu^{n}_{BF(e)}(x)+\delta\pi^{n}_{BF(e)}(x)+\nu^{p}_{BF(e)}(x)+\delta\pi^{p}_{BF(e)}(x))\}$$

$$F_{\eta,\delta}(BG, E)=\{x,(\mu^{n}_{BG(e)}(x)+\eta\pi^{n}_{BG(e)}(x)+\mu^{p}_{BG(e)}(x)+\eta\pi^{p}_{BG(e)}(x)),$$

$$(\nu^{n}_{BG(e)}(x)+\delta\pi^{n}_{BG(e)}(x), \nu^{p}_{BG(e)}(x)+\delta\pi^{p}_{BG(e)}(x))\}$$

$$F_{\eta,\delta}(BF, E)\,@\,F_{\eta,\delta}(BG, E)=\{x,(\frac{1}{2}(\mu^{n}_{BF(e)}(x)+\eta\pi^{n}_{BF(e)}(x)+\mu^{n}_{BG(e)}(x)$$

$$+\eta\pi^{n}_{BG(e)}(x)), \mu^{p}_{BF(e)}(x)+\eta\pi^{p}_{BF(e)}(x)+\mu^{p}_{BG(e)}(x)+\eta\pi^{p}_{BG(e)}(x))),$$

$$(\frac{1}{2}(\nu^{n}_{BF(e)}(x)+\delta\pi^{n}_{BF(e)}(x)+\nu^{n}_{BG(e)}(x)+\delta\pi^{n}_{BG(e)}(x)),$$

$$\upsilon^{p}_{BF(e)}(x)+\delta\pi^{p}_{BF(e)}(x)+\nu^{p}_{BG(e)}(x)+\delta\pi^{p}_{BG(e)}(x)))\}.$$

Therefore $F_{\gamma,\delta}((BF, E)\,@\,(BG, E))=F_{\gamma,\delta}(BF, E)\,@\,F_{\gamma,\delta}(BG, E)$

(iii) $G_{\eta,\delta}((BF, E)\,@\,(BG, E))$

$$=\{x,\frac{1}{2}(\eta\mu^{n}_{BF(e)}(x)+\eta\mu^{n}_{BG(e)}(x)+\eta\mu^{p}_{BF(e)}(x)+\eta\pi^{p}_{BG(e)}(x))),$$

$$(\frac{1}{2}(\delta\nu^{n}_{BF(e)}(x)+\delta\nu^{n}_{BG(e)}(x)+\delta\nu^{p}_{BF(e)}(x)+\delta\nu^{n}_{BG(e)}(x))\}$$

$$G_{\eta,\delta}(BF, E)=\{x,(\eta\mu^{p}_{BF(e)}(x)+\eta\pi^{p}_{BF(e)}(x)),(\delta\mu^{p}_{BF(e)}(x)+\delta\nu^{p}_{BF(e)}(x))\}$$

$$G_{\eta,\delta}(BG, E)=\{x,(\eta\mu^{p}_{BG(e)}(x)+\eta\mu^{p}_{BG(e)}(x)),(\delta\mu^{p}_{BG(e)}(x)+\delta\nu^{p}_{BG(e)}(x))\}$$

$$G_{\eta.\delta}(BF, E)\,@\,G_{\eta.\delta}(BG, E)$$

$$=\{x,\frac{1}{2}(\eta\mu^{n}_{BF(e)}(x)+\eta\mu^{n}_{BG(e)}(x), \eta\mu^{p}_{BF(e)}(x)+\eta\pi^{p}_{BG(e)}(x))),$$

$$(\frac{1}{2}(\delta\nu^{n}_{BF(e)}(x)+\delta\nu^{n}_{BG(e)}(x)+\delta\nu^{p}_{BF(e)}(x)+\delta\nu^{n}_{BG(e)}(x))\}.$$

Therefore $G_{\eta,\delta}((BF, E)\,@\,(BG, E))=G_{\eta,\delta}(BF, E)\,@\,G_{\eta,\delta}(BG, E)$.

3. INFORMATION MEASURE ON BIFSS

In this section, we define information measure on BIFSS and prove that it is an entropy.

Definition 3.1. Let the universal set be $U = \{A_1, A_2, \ldots, A_m\}$ and the set of parameters be $E = \{e_1, e_2, \ldots, e_n\}$. For any *BIFSS* (BF, E) an information measure to indicate the degree of fuzziness of (BF, E) is defined as

$$BI_m(BF, E_p) = \frac{1}{n}\sum_{p=1}^{n} \frac{\wedge\{\mu^n_{BF(e)}(x_p), \nu^n_{BF(e)}(x_p)\} - \wedge\{\mu^p_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p)\}}{\vee\{\mu^n_{BF(e)}(x_p), \nu^n_{BF(e)}(x_p)\} - \vee\{\mu^p_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p)\}}$$

where, $\wedge$, $\vee$ denotes the minimum and maximum, respectively.

Theorem 3.2. *The information measure $BI_m(BF, E)$ for BIFSS is an entropy.*

Proof. If $(BF, E) \in P(U)$, then for each $x_p \in A_p$ and $e \in E$ we have:

Case (i). If $(\mu^n_{BF(e)}(x_p), \mu^p_{BF(e)}(x_p)) = (\nu^n_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p))$, then

$$\mu^n_{BF(e)}(x_p) = \nu^n_{BF(e)}(x_p), \mu^p_{BF(e)}(x_p) = \nu^p_{BF(e)}(x_p)$$

$$\wedge\{\mu^n_{BF(e)}(x_p), \nu^n_{BF(e)}(x_p)\} - \wedge\{\mu^p_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p)\}$$

$$= \vee\{\mu^n_{BF(e)}(x_p), \nu^n_{BF(e)}(x_p)\} - \vee\{\mu^p_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p)\}.$$

So, $BI_m(BF, E) = 1$.

Conversely, suppose that $\mathrm{BI}_m(BF, E) = 1$.

This implies that

$$\wedge\{\mu^n_{BF(e)}(x_p), \nu^n_{BF(e)}(x_p)\} = \wedge\{\mu^n_{BF(e)}(x_p), \nu^n_{BF(e)}(x_p)\}$$

and

$$\wedge\{\mu^p_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p)\} - \wedge\{\mu^p_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p)\}.$$

Hence $\mu^n_{F(e)}(x_p) = \nu^n_{F(e)}(x_p)$ and $\mu^p_{F(e)}(x_p) = \nu^p_{F(e)}(x_p)$. Therefore, we have

$$(\mu^n_{BF(e)}(x_p), \mu^p_{BF(e)}(x_p)) = (\nu^n_{BF(e)}(x_p), \nu^p_{BF(e)}(x_p)).$$

Case (ii). Suppose that (BF, E) is less than (BG, E) then,

$$(\mu^{n}_{BF\,(e)}(x_p),\ \mu^{p}_{BF\,(e)}(x_p)) \le (\mu^{n}_{BG\,(e)}(x_p),\ \mu^{p}_{BG\,(e)}(x_p)),$$

$$(\nu^{n}_{BF\,(e)}(x_p),\ \nu^{p}_{BF\,(e)}(x_p)) \ge (\nu^{n}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)).$$

For $(\mu^{n}_{BG\,(e)}(x_p),\ \mu^{p}_{BG\,(e)}(x_p)) \le (\nu^{n}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)),$

$$\mu^{n}_{BF\,(e)}(x_p) \le \mu^{n}_{BG\,(e)}(x_p)) \le \nu^{n}_{BG\,(e)}(x_p) \le \nu^{n}_{BF\,(e)}(x_p)$$

and

$$\mu^{p}_{BF\,(e)}(x_p) \le \mu^{p}_{BG\,(e)}(x_p) \le \nu^{p}_{BG\,(e)}(x_p) \le \nu^{p}_{BF\,(e)}(x_p).$$

So,

$$\frac{\wedge\{\mu^{n}_{BF\,(e)}(x_p),\ \nu^{n}_{BF\,(e)}(x_p)\} - \wedge\{\mu^{p}_{BF\,(e)}(x_p),\ \nu^{p}_{BF\,(e)}(x_p)\}}{\vee\{\mu^{n}_{BF\,(e)}(x_p),\ \nu^{n}_{BF\,(e)}(x_p)\} - \vee\{\mu^{p}_{BF\,(e)}(x_p),\ \nu^{p}_{BF\,(e)}(x_p)\}},$$

$$= \frac{\mu^{n}_{BF\,(e)}(x_p) + \mu^{p}_{BF\,(e)}(x_p)}{\nu\mu^{n}_{BF\,(e)}(x_p) - \nu^{n}_{BF\,(e)}(x_p)} \le \frac{\mu^{n}_{BG\,(e)}(x_p) + \mu^{p}_{BG\,(e)}(x_p)}{\nu\mu^{n}_{BG\,(e)}(x_p) - \nu^{n}_{BG\,(e)}(x_p)},$$

$$= \frac{\wedge\{\mu^{n}_{BG\,(e)}(x_p),\ \nu^{n}_{BG\,(e)}(x_p)\} - \wedge\{\mu^{p}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)\}}{\vee\{\mu^{n}_{BG\,(e)}(x_p),\ \nu^{n}_{BG\,(e)}(x_p)\} - \vee\{\mu^{p}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)\}}.$$

By taking summation on both sides and dividing it by n, we get

$$\frac{1}{n}\sum_{p=1}^{n} \frac{\wedge\{\mu^{n}_{BF\,(e)}(x_p),\ \nu^{n}_{BF\,(e)}(x_p)\} - \wedge\{\mu^{p}_{BF\,(e)}(x_p),\ \nu^{p}_{BF\,(e)}(x_p)\}}{\vee\{\mu^{n}_{BF\,(e)}(x_p),\ \nu^{n}_{BF\,(e)}(x_p)\} - \vee\{\mu^{p}_{BF\,(e)}(x_p),\ \nu^{p}_{BF\,(e)}(x_p)\}}$$

$$\le \frac{1}{n}\sum_{p=1}^{n} \frac{\wedge\{\mu^{n}_{BG\,(e)}(x_p),\ \nu^{n}_{BG\,(e)}(x_p)\} - \wedge\{\mu^{p}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)\}}{\vee\{\mu^{n}_{BG\,(e)}(x_p),\ \nu^{n}_{BG\,(e)}(x_p)\} - \vee\{\mu^{p}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)\}}$$

i.e., $BI_m(BF, E) \le BI_m(BG, E)$.

Case (iii). Suppose that (BF, E) is greater than (BG, E) then,

$$(\nu^{n}_{BF\,(e)}(x_p),\ \nu^{p}_{BF\,(e)}(x_p)) \ge (\nu^{n}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)).$$

For $(\mu^{n}_{BF\,(e)}(x_p),\ \mu^{p}_{BF\,(e)}(x_p)) \ge (\nu^{n}_{BG\,(e)}(x_p),\ \nu^{p}_{BG\,(e)}(x_p)),$

$$\mu^{n}_{BF\,(e)}(x_p) \ge \mu^{n}_{BF\,(e)}(x_p)) \ge \nu^{n}_{BG\,(e)}(x_p),\ \nu^{n}_{BG\,(e)}(x_p)$$

and

$$\mu^{p}_{BF(e)}(x_p) \geq \mu^{p}_{BG(e)}(x_p)) \geq \nu^{p}_{BG(e)}(x_p) \geq \nu^{p}_{BF(e)}(x_p).$$

So,

$$\frac{\wedge\{\mu^{n}_{BF(e)}(x_p), \nu^{n}_{BF(e)}(x_p)\} - \wedge\{\mu^{p}_{BF(e)}(x_p), \nu^{p}_{BF(e)}(x_p)\}}{\vee\{\mu^{n}_{BF(e)}(x_p), \nu^{n}_{BF(e)}(x_p)\} - \vee\{\mu^{p}_{BF(e)}(x_p), \nu^{p}_{BF(e)}(x_p)\}},$$

$$= \frac{\mu^{n}_{BF(e)}(x_p) + \mu^{p}_{BF(e)}(x_p)}{\nu\mu^{n}_{BF(e)}(x_p) - \nu^{p}_{BF(e)}(x_p)} \geq \frac{\mu^{n}_{BG(e)}(x_p) - \mu^{p}_{BG(e)}(x_p)}{\nu\mu^{n}_{BG(e)}(x_p) - \nu^{p}_{BG(e)}(x_p)},$$

$$= \frac{\wedge\{\mu^{n}_{BG(e)}(x_p), \nu^{n}_{BG(e)}(x_p)\} - \wedge\{\mu^{p}_{BG(e)}(x_p), \nu^{p}_{BG(e)}(x_p)\}}{\vee\{\mu^{n}_{BG(e)}(x_p), \nu^{n}_{BG(e)}(x_p)\} - \vee\{\mu^{p}_{BG(e)}(x_p), \nu^{p}_{BG(e)}(x_p)\}}.$$

By taking summation on both sides and dividing it by n, we get

$$\frac{1}{n}\sum_{p=1}^{n}\frac{\wedge\{\mu^{n}_{BF(e)}(x_p), \nu^{n}_{BF(e)}(x_p)\} - \wedge\{\mu^{p}_{BF(e)}(x_p), \nu^{p}_{BF(e)}(x_p)\}}{\vee\{\mu^{n}_{BF(e)}(x_p), \nu^{n}_{BF(e)}(x_p)\} - \vee\{\mu^{p}_{BF(e)}(x_p), \nu^{p}_{BF(e)}(x_p)\}}$$

$$\leq \frac{1}{n}\sum_{p=1}^{n}\frac{\wedge\{\mu^{n}_{BG(e)}(x_p), \nu^{n}_{BG(e)}(x_p)\} - \wedge\{\mu^{p}_{BG(e)}(x_p), \nu^{p}_{BG(e)}(x_p)\}}{\vee\{\mu^{n}_{BG(e)}(x_p), \nu^{n}_{BG(e)}(x_p)\} - \vee\{\mu^{p}_{BG(e)}(x_p), \nu^{p}_{BG(e)}(x_p)\}}$$

i.e., $BI_m(BF, E) \geq BI_m(BG, E)$.

Case (iv).

$$(BF, E)^c = \{x, (\nu^{n}_{BF(e)}(x_p), \nu^{p}_{BF(e)}(x_p))$$

$$(\mu^{n}_{BF(e)}(x_p), \mu^{n}_{BF(e)}(x_p)) : x_p \in A_p, e \in E\}$$

Therefore,

$$BI_m(BF, E)^C = \frac{1}{n}\sum_{p=1}^{n}\frac{\wedge\{\nu^{n}_{BF(e)}(x_p), \mu^{n}_{BF(e)}(x_p)\} - \wedge\{\nu^{p}_{BF(e)}(x_p), \mu^{p}_{BF(e)}(x_p)\}}{\vee\{\nu^{n}_{BF(e)}(x_p), \mu^{n}_{BF(e)}(x_p)\} - \vee\{\nu^{p}_{BF(e)}(x_p), \mu^{p}_{BF(e)}(x_p)\}}$$

$$= BI_m(BF, E).$$

In this section, we develop a decision making concept based on information measure of BIFSS. A procedure for decision making method is framed.

4.1. Procedure:

The following are the steps to be followed for this method:

Step 1: Construct the *BIFS* sets (BF, E_q) over U.

Step 2. Determine the information measure of (BF, E_q) by using

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Definition 3.1.

Step 3. Compare the values of $BI_m(BF, E_q)$ and conclude. Thus the better choice is the alternative for which the information measure is the least.

Example 4.2. A customer decides to buy a television. Television (alternatives) of three different companies (BF, E_1), (BF, E_2), (BF, E_3) are evaluated over six factors $\{t_1, t_2, t_3, t_4, t_5, t_6\}$, where t_1 = price range, t_2 LED or $OLED$, t_3 = cables and accessories, t_4 = audio upgrades, t_5 = screen size and t_6 = smart television. Depending on the six factors, the best alternative is determined.

Step 1. BIFSS over U to assess the alternatives is as follows:

U	(BF, E_1)	(BF, E_2)	(BF, E_3)
t_1	((-0.13,0.47)(-0.25,0.1))	((-0.2,0.68),(-0.32,0.15))	((0.18,0.42),(0.28,0.12))
t_2	((-0.34,0.62),(-0.16,0.28))	((-0.18,0.83),(-0.61,0.1))	((0.17,0.22),(0.18,0.06))
t_3	((-0.25,0.73),(-0.4,0.14))	((-0.32,0.45),(-0.27,0.13))	((-0.14,0.73),(0.3,0.12))
t_4	((-0.1,0.51),(-0.38,0.08))	((-0.41,0.64),(-0.5,0.3))	((-0.24,0.6),(0.04,0.11))
t_5	((-0.12,0.38),(-0.26,0.04))	((-0.24,0.76),(-0.61,0.15))	((0.31,0.52),(0.17,0.28))
t_6	((-0.27,0.47), (-0.3,0.13))	((-0.3,0.54),(-0.43,0.28))	((0.16,0.72),(0.22,0.12))

Step 2. The $BI_m(BF, E_q)$ are estimated as follows:

$$BI_m(BF, E_q) = 0.6987$$

$$BI_m(BF, E_2) = 0.6969$$

$$BI_m(BF, E_3) = 0.5922 \quad .$$

Step 3. The best alternative is the one which has least information measure with respect to six factors. We have $BI_m(BF, E_3) < BI_m(BF, E_2) < BI_m(BF, E_1)$. Therefore, the information measure $BI_m(BF, E_3)$ has the least value and its corresponding alternative is the best alternative. So, the television (BF, E_3) is the best one among the others.

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Application of Intuitionistic Fuzzy Rough Matrices

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ABSTRACT

Rough set introduced by Pawlak is a mathematical tool to deal with a special type of uncertainty. In this a set with incomplete and insufficient information is represented by set approximations called the lower and upper approximations. Intuitionistic fuzzy rough set is a generalization obtained by combining the notion of intuitionistic fuzzy set and rough set. Intuitionistic fuzzy rough matrices arise when a finite number of intuitionistic fuzzy rough sets are defined over a finite universe. The purpose of this paper is to define intuitionistic fuzzy rough matrices and study some of their theoretical properties including some operations between them. Decision making based on composition of intuitionistic fuzzy rough matrices is developed. An example is presented to illustrate the working of the method.

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Keywords: Intuitionistic fuzzy rough matrix, operations on intuitionistic fuzzy rough matrices, decision making technique.

1. INTRODUCTION

The theory of rough sets was initiated by Pawlak [4, 5]. It is an extension of classical set theory for the study of systems characterized by insufficient and incomplete information. A key notion in Pawlak rough set model is the equivalence relation, where equivalence classes serve as the building blocks for the construction of lower and upper approximations. Replacing the equivalence relation by an arbitrary binary relation, different kinds of generalizations in Pawlak rough set models were obtained.

Matrices play an important role in the broad area of science and engineering. The classical matrix theory cannot answer questions involving various types of uncertainties. Yang et al. [6] initiated a matrix representation of fuzzy soft set and studied their basic properties. They defined products of fuzzy soft matrices that satisfy commutative law and used them in a decision making method. The notion of fuzzy soft matrix was studied by Borah et al. [1].

In [7] Yang et al. introduced the concept of interval-valued fuzzy soft set. They defined complement, AND, OR operations and proved De Morgan's, associative and distributive laws for the interval-valued fuzzy soft sets. They also developed a decision making method using interval valued fuzzy soft sets. Some numerical examples are employed to substantiate the theoretical arguments.

Lei Zhou et al. [3] proposed a general framework for the study of relation based intuitionistic fuzzy rough approximation operators within which both constructive and axiomatic approaches were used and established some basic properties of intuitionistic fuzzy rough approximation operators. Chetia et al. [2] defined intuitionistic fuzzy soft matrices and some operations on these matrices.

In this paper we define intuitionistic fuzzy rough matrix and and study some of their theoretical properties including some operations between them. Decision making method based on composition of intuitionistic fuzzy rough matrices is developed. Numerical example is given to illustrate the application of the method.

2. Preliminaries

Definition 2. 1 [3]. Let U be a nonempty and finite universe of discourse.

An intuitionistic fuzzy relation IFR on U is an intuitionistic fuzzy subset of U × U , viz.,

$$\overline{IF} = \{\langle (x, y), \mu(x, y), \upsilon(x, y) \mid (x, y) \in U \times U \rangle\}$$

Where $\mu : U \times U \to [0, 1]$ and $\upsilon : U \times U \to [0, 1]$ denote the membership and non membership values of (x, y) satisfying the condition $0 \le \mu(x, y) + \upsilon(x, y) \le 1$ for all $(x, y) \in U \times U$.

Definition 2.2 [3]. Let IFR be an intuitionistic fuzzy relation defined on (U × U). The pair (U IF R) & , is called an intuitionistic fuzzy rough approximation space. For any A ∈ IF (U), where I F (U) denotes the intuitionistic fuzzy power set of U, the lower and upper approximations of A with respect to (U , IFR) denoted by IF R (A) and IF R (A) are defined as follows:

$$IF\ \underline{R}(A) = \{x, \mu_{IF\ \underline{R}(A)}(x), \upsilon_{IF\ \underline{R}(A)}(x)/x \in U\}$$

$$IF\ \overline{R}(A) = \{x, \mu_{IF\overline{R}(A)}(x), \upsilon_{IF\overline{R}(A)}(x)/x \in U\},$$

Where

$$\mu_{IF\ \underline{R}(A)}(x) = \wedge_{y \in U} [\upsilon_{IF\ \underline{R}}(x, y) \vee \mu_A(y)],$$
$$\upsilon_{IF\ \underline{R}(A)}(x) = \vee_{y \in U} [\mu_{IF\ \underline{R}}(x, y) \wedge \upsilon_A(y)].$$
$$\mu_{IF\ \overline{R}(A)}(x) = \vee_{y \in U} [\mu_{IF\overline{R}}(x, y) \wedge \mu_A(y)],$$
$$\upsilon_{IF\ \overline{R}(A)}(x) = \wedge_{y \in U} [\upsilon_{IF\overline{R}}(x, y) \vee \upsilon_A(y)].$$

The pair $(IF\ \underline{R}(A), IF\ \overline{R}(A))$ is called the intuitionistic fuzzy rough set associated A denoted by $IFR\ (A)$.

Example 2.3. Let (U, IFR) be an intuitionistic fuzzy rough approximation space, where $U = \{x_1, x_2, x_3\}$ and

$$A = \{\langle x_1, 0.9, 0.08\rangle, \langle x_2, 1, 0\rangle, \langle x_3, 0.7, 0.3\rangle\},$$

$$IF\ \overline{R} = \{\langle (x_1, x_1), 0.9, 0.1\rangle, \langle (x_1, x_2), 0.3, 0.6\rangle, \langle (x_1, x_3), 0.4, 0.4\rangle$$
$$\langle (x_2, x_1), 0.4, 0.6\rangle, \langle (x_2, x_2), 1, 0\rangle, \langle (x_2, x_3), 0.7, 0.3\rangle,$$
$$\langle (x_3, x_1), 0.2, 0.5\rangle, \langle (x_3, x_2), 0.5, 0.5\rangle, \langle (x_3, x_3), 1, 0\rangle\}.$$

Then

$$\mu_{IF\ \underline{R}(A)}(x_1) = 0.7, \mu_{IF\ \underline{R}(A)}(x_2) = 0.7, \mu_{IF\ \underline{R}(A)}(x_3) = 0.7, \upsilon_{IF\ \underline{R}(A)}(x_1) = 0.3,$$
$$\upsilon_{IF\ \underline{R}(A)}(x_2) = 0.3, \upsilon_{IF\ \underline{R}(A)}(x_3) = 0.3,$$
$$\mu_{IF\overline{R}(A)}(x_1) = 0.9, \mu_{IF\overline{R}(A)}(x_2) = 1, \mu_{IF\overline{R}(A)}(x_3) = 0.7, \upsilon_{IF\overline{R}(A)}(x_1) = 0.1,$$
$$\upsilon_{IF\overline{R}(A)}(x_2) = 0, \upsilon_{IF\overline{R}(A)}(x_3) = 0.3.$$

Hence

$IFR\ (A) = \{\{\langle x_1, 0.7, 0.3\rangle, \langle x_2, 0.7, 0.3\rangle, \langle x_3, 0.7, 0.3\rangle\}, \{\langle x_1, 0.9, 0.1\rangle, \langle x_2, 1, 0\rangle, \langle x_3, 0.7, 0.3\rangle\}\}.$

Definition 2.4. Let $U = \{x_1, x_2, \ldots, x_p\}$ be the universal set and $IFR\ (A_j), j = 1, 2, \ldots, q$ denote the intuitionistic fuzzy rough sets defined on U. An intuitionistic fuzzy rough matrix associated with U and $\{A_j\}$ is a $p \times q$ matrix expressed as

$$IFRM = (IF\ \underline{R}M,\ IF\ \overline{R}M)_{p\times q},$$

where $IFRM = ifrm_{ij}, i = 1, 2, \ldots, p\ j = 1, 2, \ldots, q$ and $ifrm_{ij} = ((\mu_{if\underline{r}m_{ij}}, \mu_{if\overline{r}m_{ij}}), (v_{if\underline{r}m_{ij}}, v_{if\overline{r}m_{ij}}))\ \forall i, j \cdot \mu_{if\underline{r}m_{ij}}, \mu_{if\overline{r}m_{ij}}$ represent the lower and upper approximations of the degree of membership and $v_{if\underline{r}m_{ij}}, v_{if\overline{r}m_{ij}}$ represent the lower and upper approximations of the degree of non-membership satisfying the conditions $\mu_{if\overline{r}m_{ij}} + v_{if\overline{r}m_{ij}} \le 1$ for all i, j.

Definition 2.5. An intuitionistic fuzzy rough matrix of order p × q is called intuitionistic fuzzy rough null matrix if all its elements are ((0 , 0) (1, 1)). It is denoted by IFR ϕ.

Definition 2.6. An intuitionistic fuzzy rough matrix of order p × q is called intuitionistic fuzzy rough absolute matrix if all its elements are ((1, 1) (0 , 0)). It is denoted by IFRA.

Definition 2.7. Two intuitionistic fuzzy rough matrices $IFRM = ifrm_{ij}$ and $IFRN = ifrn_{ij}$ of the same order are equal if and only if $\mu_{if\underline{r}m_{ij}} = \mu_{if\underline{r}n_{ij}}, \mu_{if\overline{r}m_{ij}} = \mu_{if\overline{r}n_{ij}}$ and $v_{if\underline{r}m_{ij}} = v_{if\underline{r}n_{ij}}, v_{if\overline{r}m_{ij}} = v_{if\overline{r}n_{ij}}$ for all i, j.

3. OPERATIONS ON INTUITIONISTIC FUZZY ROUGH MATRICES

Definition 3.1. The sum of two intuitionistic fuzzy rough matrices IFRM and IFRN of the same order is defined as

$$IFRM + IFRN = [\max\ (\mu_{if\underline{r}m_{ij}}, \mu_{if\underline{r}n_{ij}}), \max\ (\mu_{if\overline{r}m_{ij}}, \mu_{if\overline{r}n_{ij}})]$$
$$= [\min\ (v_{if\underline{r}m_{ij}}, v_{if\underline{r}n_{ij}}), \min\ (v_{if\overline{r}m_{ij}}, v_{if\overline{r}n_{ij}})].$$

Example 3.2. The intuitionistic fuzzy rough matrix associated with $U = \{X_1, X_2, X_3\}$ and $\{A_1, A_2, A_3\}$ is given by

$$IFRM = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} (0.6, 0.9)(0.4, 0.1) & (0.6, 0.75)(0.4, 0.25) & (0.7, 0.75)(0.3, 0.25) \\ (0.7, 0.8)(0.3, 0.2) & (0.7, 0.7)(0.3, 0.3) & (0.7, 0.7)(0.3, 0.29) \\ (0.76, 0.9)(0.14, 0.1) & (0.76, 0.81)(0.16, 0.16) & (0.8, 0.88)(0.16, 0.09) \end{pmatrix} \end{array}$$

For the same universal set $U = \{X_1, X_2, X_3\}$ the intuitionistic fuzzy rough matrix associated with $\{B_1, B_2, B_3\}$ is given by

$$
\begin{array}{cc}
 & \begin{array}{ccc} \qquad B_1 \qquad & \qquad B_2 \qquad & \qquad B_3 \qquad \end{array} \\
IFRM = \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} & \begin{pmatrix} (0.5, 0.8)(0.5, 0.2) & (0.8, 0.8)(0.19, 0.19) & (0.5, 0.8)(0.5, 0.19) \\ (0.6, 0.6)(0.4, 0.4) & (0.75, 0.79)(0.25, 0.15) & (0.6, 0.84)(0.4, 0.16) \\ (0.75, 0.8)(0.25, 0.2) & (0.75, 0.9)(0.25, 0.1) & (0.8, 0.89)(0.2, 0.11) \end{pmatrix}
\end{array}
$$

Now

$$
IFRM + IFRN = \begin{pmatrix} (0.6, 0.9)(0.4, 0.1) & (0.8, 0.8)(0.19, 0.19) & (0.7, 0.8)(0.3, 0.19) \\ (0.7, 0.8)(0.3, 0.2) & (0.75, 0.79)(0.25, 0.15) & (0.7, 0.84)(0.3, 0.16) \\ (0.76, 0.9)(0.14, 0.1) & (0.76, 0.9)(0.16, 0.1) & (0.81, 0.89)(0.16, 0.09) \end{pmatrix}.
$$

Definition 3.3. The difference between two intuitionistic fuzzy rough matrices IFRM and IFRN of the same order is defined as

$$
IFRM - IFRN = [\min(\mu_{if\,\underline{r}m_{ij}}, \mu_{if\,\underline{r}n_{ij}}), \min(\mu_{if\,\overline{r}m_{ij}}, \mu_{if\,\overline{r}n_{ij}})]
$$

$$
= [\max(\upsilon_{if\,\underline{r}m_{ij}}, \upsilon_{if\,\underline{r}n_{ij}}), \max(\upsilon_{if\,\overline{r}m_{ij}}, \upsilon_{if\,\overline{r}n_{ij}})].
$$

for all i, j.

Example 3.4. Consider the intuitionistic fuzzy rough matrices $IFRM$ and $IFRN$ as in Example 3.2.

$$
IFRM - IFRN = \begin{pmatrix} (0.5, 0.8)(0.5, 0.2) & (0.6, 0.75)(0.4, 0.25) & (0.5, 0.75)(0.5, 0.25) \\ (0.6, 0.6)(0.4, 0.4) & (0.7, 0.7)(0.3, 0.3) & (0.6, 0.7)(0.4, 0.29) \\ (0.75, 0.8)(0.25, 0.2) & (0.75, 0.81)(0.25, 0.16) & (0.8, 0.88)(0.2, 0.11) \end{pmatrix}
$$

Definition 3.5. The AND operator between two intuitionistic fuzzy rough matrices IFRM and IFRN denoted by IFRN $\wedge$ IFRN is the intuitionistic fuzzy rough matrix

$$
IFRM \wedge IFRN = (ifrm_{ij}, ifrn_{ij})_{p^2 \times q}\ \forall i, j = 1, 2, \ldots, p
$$

and

$$
ifrm_{ij} \times ifrn_{ij} = [\min(\mu_{if\,\underline{r}m_{ij}}, \mu_{if\,\underline{r}n_{ij}}), \min(\mu_{if\,\overline{r}m_{ij}}, \mu_{if\,\overline{r}n_{ij}})]
$$

$$
[\max(\upsilon_{if\,\underline{r}m_{ij}}, \upsilon_{if\,\underline{r}n_{ij}}), \max(\upsilon_{if\,\overline{r}m_{ij}}, \upsilon_{if\,\overline{r}n_{ij}})], i, j = 1, 2, \ldots, p.
$$

Example 3.6. Consider the intuitionistic fuzzy rough matrices IFRM and IFRN as in Example 3.2.

$$
IFRM \wedge IFRN = (ifrm_{ij}, ifrn_{ij})_{p^2 \times q}\ \forall i, j = 1, 2, \ldots, p
$$

$$
\begin{pmatrix} (0.5, 0.8)(0.5, 0.2) & (0.6, 0.75)(0.4, 0.25) & (0.5, 0.75)(0.5, 0.25) \\ (0.6, 0.6)(0.4, 0.4) & (0.6, 0.75)(0.4, 0.25) & (0.6, 0.75)(0.4, 0.25) \\ (0.6, 0.8)(0.4, 0.2) & (0.6, 0.75)(0.4, 0.25) & (0.7, 0.75)(0.3,. 0.25) \\ (0.5, 0.8)(0.5, 0.2) & (0.7, 0.7)(0.3, 0.3) & (0.5, 0.7)(0.5, 0.29) \\ (0.6, 0.6)(0.4, 0.4) & (0.7, 0.7)(0.3, 0.3) & (0.6, 0.7)(0.4, 0.29) \\ (0.7, 0.8)(0.3, 0.2) & (0.7, 0.7)(0.3, 0.3) & (0.7, 0.7)(0.3, 0.29) \\ (0.5, 0.8)(0.5, 0.2) & (0.76, 0.8)(0.19, 0.19) & (0.5, 0.8)(0.5, 0.19) \\ (0.6, 0.6)(0.4, 0.4) & (0.75, 0.79)(0.25, 0.16) & (0.6, 0.84)(0.4, 0.16) \\ (0.75, 0.8)(0.25, 0.2) & (0.75, 0.81)(0.25, 0.16) & (0.8, 0.88)(0.2, 0.11) \end{pmatrix}
$$

Definition 3.7. The OR operator between two intuitionistic fuzzy rough matrices IFRM and IFRN denoted by IFRM $\vee$ IFRN is the intuitionistic fuzzy rough matrix

$$IFRM \vee IFRN = (ifrm_{ij}, ifrn_{ij})_{p^2 \times q} \; \forall i, j = 1, 2, \ldots, p$$

and

$$ifrm_{ij} \times ifrn_{ij} = [\max(\mu_{if\underline{r}m_{ij}}, \mu_{if\underline{r}n_{ij}}), \max(\mu_{if\overline{r}m_{ij}}, \mu_{if\overline{r}n_{ij}})]$$

$$[\min(\upsilon_{if\underline{r}m_{ij}}, \upsilon_{if\underline{r}n_{ij}}), \min(\upsilon_{if\overline{r}m_{ij}}, \upsilon_{if\overline{r}n_{ij}})], \; i, j = 1, 2, \ldots, p.$$

Example 3.8. Consider the intuitionistic fuzzy rough matrices IFRM and IFRN as in Example 3.2.

$$IFRM \vee IFRN = (ifrm_{ij}, ifrn_{ij})_{p^2 \times q} \; \forall i, j = 1, 2, \ldots, p$$

$$\begin{pmatrix} (0.6, 0.9)(0.4, 0.1) & (0.8, 0.8)(0.19, 0.19) & (0.7, 0.8)(0.3, 0.19) \\ (0.6, 0.9)(0.4, 0.1) & (0.75, 0.79)(0.25, 0.15) & (0.7, 0.84)(0.3, 0.16) \\ (0.75, 0.9)(0.25, 0.1) & (0.75, 0.9)(0.25, 0.1) & (0.8, 0.79)(0.2, 0.11) \\ (0.7, 0.8)(0.3, 0.2) & (0.8, 0.8)(0.19, 0.19) & (0.7, 0.8)(0.3, 0.19) \\ (0.75, 0.8)(0.3, 0.2) & (0.75, 0.79)(0.25, 0.15) & (0.7, 0.84)(0.3, 0.16) \\ (0.75, 0.8)(0.25, 0.2) & (0.75, 0.9)(0.25, 0.1) & (0.8, 0.89)(0.2, 0.11) \\ (0.76, 0.9)(0.14, 0.1) & (0.8, 0.81)(0.16, 0.16) & (0.81, 0.88)(0.16, 0.09) \\ (0.76, 0.9)(0.14, 0.1) & (0.76, 0.81)(0.16, 0.15) & (0.81, 0.88)(0.16, 0.09) \\ (0.76, 0.9)(0.14, 0.1) & (0.76, 0.9)(0.16, 0.1) & (0.81, 0.89)(0.16, 0.09) \end{pmatrix}.$$

Definition 3.9. The max-min composition of the two intuitionistic fuzzy rough matrices IFRM $p \times q$ and IFRN $p \times q$ denoted by IFRM $*$ IFRN is defined as

$$IFRM * IFRN = IFRD_{p \times r} = [\max_j \min(\mu_{if\underline{r}m_{ij}}, \mu_{if\underline{r}n_{jk}}), \max_j \min(\mu_{if\overline{r}m_{ij}}, \mu_{if\overline{r}n_{jk}})]$$

$$[\min_j \max(\upsilon_{if\underline{r}m_{ij}}, \upsilon_{if\underline{r}n_{jk}}), \min_j \max(\upsilon_{if\overline{r}m_{ij}}, \upsilon_{if\overline{r}n_{jk}})] \; \forall i, j, k.$$

Definition 3.10. The intuitionistic fuzzy rough complement of IFRM denoted by IFRM is defined as IFR $M' = (ifrm_{ij})$, where

$$ifrm'_{ij} = ((\upsilon_{if\underline{r}m_{ij}}, \upsilon_{if\overline{r}m_{ij}}), (\mu_{if\underline{r}m_{ij}}, \mu_{if\overline{r}m_{jk}})) \; \forall i, j$$

where

$$\mu_{if\overline{r}m_{ij}} + \upsilon_{if\overline{r}m_{ij}} \leq 1.$$

Theorem 3.11. Commutative law holds for intuitionistic fuzzy rough matrices over the universe U.

(i) $IFRM \wedge IFRN = IFRN \wedge IFRM$

(ii) $IFRM \vee IFRN = IFRN \vee IFRM$.

Proof.

(i) $IFRM \wedge IFRN = \min(\mu_{ifrm_{ij}}, \mu_{ifrn_{ij}}), \max(\upsilon_{ifrm_{ij}}, \upsilon_{ifrn_{ij}})$

$= \min(\mu_{ifrn_{ij}}, \mu_{ifrm_{ij}}), \max(\upsilon_{ifrn_{ij}}, \upsilon_{ifrm_{ij}})$

$= IFRN \wedge IFRM$.

(ii) $IFRM \vee IFRN = \max(\mu_{ifrm_{ij}}, \mu_{ifrn_{ij}}), \min(\upsilon_{ifrm_{ij}}, \upsilon_{ifrn_{ij}})$

$= \max(\mu_{ifrn_{ij}}, \mu_{ifrm_{ij}}), \min(\upsilon_{ifrn_{ij}}, \upsilon_{ifrm_{ij}})$

$= IFRN \vee IFRM$.

Theorem 3.12. *Associative law holds for intuitionistic fuzzy rough matrices over the universe U.*

(i) $(IFRM \wedge IFRN) \wedge IFRO = IFRM \wedge (IFRN \wedge IFRO)$

(ii) $(IFRM \vee IFRN) \vee IFRO = IFRM \vee (IFRN \vee IFRO)$

Proof.

(i) $(IFRM \wedge IFRN) \wedge IFRO = (\min(\mu_{ifrm_{ij}}, \mu_{ifrn_{ij}}), \max(\upsilon_{ifrm_{ij}}, \upsilon_{ifrn_{ij}})) \wedge ((\mu_{ifro_{ij}}), (\upsilon_{ifro_{ij}}))$

$= (\min(\mu_{ifrm_{ij}}, \mu_{ifrn_{ij}}, \mu_{ifro_{ij}}), \max(\upsilon_{ifrm_{ij}}, \upsilon_{ifrn_{ij}}, \upsilon_{ifro_{ij}}))$

$= ((\mu_{ifrm_{ij}}), (\upsilon_{ifrm_{ij}})) \wedge (\min(\mu_{ifrn_{ij}}, \mu_{ifro_{ij}}), \max(\upsilon_{ifrn_{ij}}, \upsilon_{ifro_{ij}}))$

$= IFRM \wedge (IFRN \wedge IFRO)$.

(ii) $(IFRM \vee IFRN) \vee IFRO = (\max(\mu_{ifrm_{ij}}, \mu_{ifrn_{ij}}), \min(\upsilon_{ifrm_{ij}}, \upsilon_{ifrn_{ij}})) \vee ((\mu_{ifro_{ij}}), (\upsilon_{ifro_{ij}}))$

$= (\max(\mu_{ifrm_{ij}}, \mu_{ifrn_{ij}}, \mu_{ifro_{ij}}), \min(\upsilon_{ifrm_{ij}}, \upsilon_{ifrn_{ij}}, \upsilon_{ifro_{ij}}))$

$= ((\mu_{ifrm_{ij}}), (\upsilon_{ifrm_{ij}})) \vee (\max(\mu_{ifrn_{ij}}, \mu_{ifro_{ij}}), \min(\upsilon_{ifrn_{ij}}, \upsilon_{ifro_{ij}}))$

$= IFRM \vee (IFRN \vee IFRO)$.

Theorem 3.13. Distributive law holds for intuitionistic fuzzy rough matrices over the universe U.

(i) $IFRM \wedge (IFRN \vee IFRO) = (IFRM \wedge IFRN) \vee (IFRM \wedge IFRO)$

(ii) $IFRM \vee (IFRN \wedge IFRO) = (IFRM \vee IFRN) \wedge (IFRM \vee IFRO)$.

Proof. Proof is straight forward.

4. APPLICATION OF INTUITIONISTIC FUZZY ROUGH MATRICES

In this section some applications of intuitionistic fuzzy rough matrices in real life situations are presented.

4.1. Statement of the problem:

Let $X = \{x_1, x_2, \dots, x_n\}$ be a given universal set and $Y = \{y_1, y_2, \dots, y_m\}$ be a given set of properties based on which the elements of the universal set are described in the form of intuitionistic fuzzy rough matrices. Let $Z = \{z_1, z_2, \dots, z_p\}$ be the set of possible conclusions that can be drawn on the individual elements of U based on the set of properties Y. The effects of the properties on the conclusion are given in the form of another intuitionistic fuzzy rough matrix of appropriate order. The problem is to arrive at the best conclusion for each element of the universal set based on the properties.

4.2. The Method

Let $X = \{x_1, x_2, \ldots, x_p\}$ be the given set of objects. Let $A = \{A_1, A_2, \ldots, A_q\}$ be the set of properties associated with the given set of objects and let $B = \{B_1, B_2, \ldots, B_r\}$ be the set of conclusions arrived on U based on A. Let $IFRM = ((\mu_{if\,\underline{r}m_{ij}}, \mu_{if\bar{r}m_{ij}})(\upsilon_{if\,\underline{r}m_{ij}}, \upsilon_{if\bar{r}m_{ij}}))$ be the intuitionistic fuzzy rough matrix associated with $U \times A$ and $IFRN = ((\mu_{if\,\underline{r}n_{jk}}, \mu_{if\bar{r}n_{ij}})(\upsilon_{if\,\underline{r}n_{ij}}, \upsilon_{if\bar{r}n_{jk}}))$ be the intuitionistic fuzzy rough matrix associated with $A \times B$. Let

$$IFRT = ((\mu_{if\,\underline{r}t_{ik}}, \mu_{if\bar{r}t_{ik}})(\upsilon_{if\,\underline{r}t_{ik}}, \upsilon_{if\bar{r}t_{ik}})) = IFRM * IFRN$$

be the composition of IFRM and IFRN. Let

$$IFRS = ((\mu_{if\,\underline{r}s_{ik}}, \mu_{if\bar{r}s_{ik}})(\upsilon_{if\,\underline{r}s_{ik}}, \upsilon_{if\bar{r}s_{ik}})) = IFRM * IFRN$$

be the composition of IFRM and IFRN, where IFRN denotes the complement of IFRN . IFRT and IFRS are p × r intuitionistic fuzzy rough matrices giving the conclusions on the elements of the universal set U based on the influence of A and B.

$$\text{Let } X_{ik} = (\max(\mu_{if\bar{r}t_{ik}}, \mu_{if\,\underline{r}s_{ik}}), \min(\upsilon_{if\bar{r}t_{ik}}, \upsilon_{if\,\underline{r}s_{ik}}))$$

$$\text{And } Y_{ik} = (\max(\mu_{if\,\underline{r}t_{ik}}, \mu_{if\,\underline{r}s_{ik}}), \min(\upsilon_{if\,\underline{r}t_{ik}}, \upsilon_{if\bar{r}s_{ik}})).$$

$$Z_{ik} = \left(\left\{\frac{\mu_{ifrx_{jk}} + \mu_{ifry_{jk}}}{2}\right\}, \left\{\frac{\upsilon_{ifrx_{jk}} - \upsilon_{ifry_{jk}}}{2}\right\}\right).$$

The elements of Zik give the maximum intuitionistic fuzzy membership and minimum intuitionistic fuzzy non membership values of each element of the universal set based on the effect of A in relation with B. It is a p × r matrix whose rows are labeled by elements of the universal set and the columns are labeled by the elements of B. To arrive at a conclusion regarding each element of the universal set, we compare between the elements of first column and conclude that the element of the column which has maximum intuitionistic fuzzy membership and minimum intuitionistic fuzzy non membership values is the best among the elements of the universal set in respect of the first element of B. Similar conclusions can be drawn with respect to other elements of B.

4.3. Algorithm

The steps of the algorithm based on the method explained above are as follows:

Step 1: Input the intuitionistic fuzzy rough matrix IFRM on U × A.

Step 2: Input the intuitionistic fuzzy rough matrix IFRN on A × B.

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Step 3: Compute the matrices IFRS = IFRM ∗ IFRN and

IFRS = IFRM ∗ IFRN .

Step 4: Compute the values X_{ik} and Y_{ik} where,

$$X_{ik} = (\max(\mu_{if\overline{r}t_{ik}}, \mu_{if\underline{r}s_{ik}}), \min(\upsilon_{if\overline{r}t_{ik}}, \upsilon_{if\underline{r}s_{ik}}))$$

$$Y_{ik} = (\max(\mu_{if\underline{r}t_{ik}}, \mu_{if\underline{r}s_{ik}}), \min(\upsilon_{if\underline{r}t_{ik}}, \upsilon_{if\overline{r}s_{ik}})) \text{ and}$$

$$Z_{ik} = \left(\left\{\frac{\mu_{ifrx_{jk}} + \mu_{ifry_{jk}}}{2}\right\}, \left\{\frac{\upsilon_{ifrx_{jk}} - \upsilon_{ifry_{jk}}}{2}\right\}\right).$$

Step 5: Compare and conclude. The entry Z_{ik} that corresponds to the maximum membership and minimum non-membership values under each column identifies the best element of the universal set with respect to that column.

4.4. Example

Suppose there are four patients P_1, P_2, P_3 and P_4 in a hospital. Let $U = \{S_1, S_2, S_3, S_4\}$ be the universal set representing the symptoms weight loss, headache, fatigue and rashes, respectively. Step 1: The intuitionistic fuzzy rough matrix IFRM representing Patients symptoms is given below:

$$IFRM = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{pmatrix} (0.6,0.8)(0.29,0.1) & (0.75,0.9)(0.14,0.07) & (0.68,0.69)(0.29,0.2) & (0.88,0.9)(0.1,0.07) \\ (0.6, 0.7)(0.4,0.29) & (0.7,0.8)(0.3,0.2) & (0.68,0.69)(0.29,0.2) & (0.73,0.88)(0.22,0.12) \\ (0.75,0.9)(0.2,0.03) & (0.78,0.87)(0.19,0.1) & (0.8,0.85)(0.12,0.08) & (0.79,0.92)(0.16,0.06) \\ (0.7, 0.9)(0.19,0.1) & (0.6,0.85)(0.2,0.08) & (0.65,0.76)(0.17,0.15) & (0.79,0.83)(0.15,0.09) \end{pmatrix} \end{array}$$

$U = \{D_1, D_2, D_3, D_4\}$ be the set of diseases under consideration.

Let D_1 = chicken pox, D_2 = small pox, D_3 = mumps and D_4 = measles.

Step 2: The intuitionistic fuzzy rough matrix IFRN representing symptoms diseases is given below:

$$IFRN = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{pmatrix} (0.65,0.9)(0.25,0.1) & (0.8,0.8)(0.15,0.15) & (0.69,0.73)(0.18,0.13) & (0.7,0.74)(0.22,0.17) \\ (0.69, 0.9)(0.3,0.1) & (0.66,0.75)(0.27,0.21) & (0.75, 0.8)(0.15,0.12) & (0.82,0.88)(0.18,0.11) \\ (0.69,0.9)(0.19,0.07) & (0.75,0.75)(0.25,0.24) & (0.77,0.8)(0.24, 0.2) & (0.81,0.82)(0.18,0.14) \\ (0.73, 0.85)(0.22,0.13) & (0.68,0.8)(0.3,0.16) & (0.79,0.9)(0.19, 0.08) & (0.81,0.85)(0.18,0.14) \end{pmatrix} \end{array}$$

$$IFRN' = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{pmatrix} (0.25,0.1)(0.65,0.9) & (0.15,0.15)(0.8,0.8) & (0.18,0.13)(0.69,0.75) & (0.2,0.17)(0.7,0.74) \\ (0.3, 0.1)(0.67,0.9) & (0.27,0.21)(0.66,0.75) & (0.15, 0.12)(0.75,0.8) & (0.18,0.11)(0.82,0.88) \\ (0.19,0.07)(0.69,0.9) & (0.25,0.24)(0.75,0.75) & (0.24,0.2)(0.77, 0.8) & (0.18,0.14)(0.81,0.82) \\ (0.22, 0.13)(0.73,0.85) & (0.3,0.16)(0.68,0.8) & (0.19,0.08)(0.79, 0.9) & (0.18,0.14)(0.81,0.85) \end{pmatrix} \end{array}$$

Step 3: The composition of the intuitionistic fuzzy rough matrices IFRM ∗ IFRN and IFRM ∗ IFRN give two intuitionistic fuzzy rough matrices IFRT and IFRS respectively, as given below:

$$IFRT = \begin{array}{cc} & \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{pmatrix} (0.73,0.9)(0.22,0.1) & (0.68,0.8)(0.27,0.15) & (0.79,0.9)(0.15,0.08) & (0.81,0.88)(0.18,0.11) \\ (0.73,0.9)(0.19,0.1) & (0.75,0.8)(0.25,0.16) & (0.77,0.8)(0.22,0.12) & (0.8,0.85)(0.18,0.14) \\ (0.73,0.9)(0.19,0.08) & (0.75,0.8)(0.2,0.15) & (0.79,0.9)(0.19,0.08) & (0.79,0.87)(0.18,0.11) \\ (0.73,0.9)(0.19,0.1) & (0.7,0.8)(0.19,0.15) & (0.79,0.83)(0.19,0.09) & (0.79,0.85)(0.18,0.11) \end{pmatrix} \end{array}$$

$$IFRS = \begin{array}{cc} & \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{pmatrix} (0.3,0.13)(0.65,0.85) & (0.3,0.24)(0.66,0.75) & (0.24,0.2)(0.69,0.75) & (0.22,0.17)(0.7,0.74) \\ (0.3,0.13)(0.65,0.85) & (0.3,0.24)(0.66,0.75) & (0.24,0.2)(0.69,0.75) & (0.22,0.17)(0.7,0.74) \\ (0.3,0.13)(0.65,0.85) & (0.3,0.24)(0.66,0.75) & (0.24,0.2)(0.69,0.75) & (0.22,0.17)(0.7,0.74) \\ (0.3,0.13)(0.65,0.85) & (0.3,0.24)(0.66,0.75) & (0.24,0.2)(0.69,0.75) & (0.22,0.17)(0.7,0.74) \end{pmatrix} \end{array}$$

Step 4: Computation of X_{ik}, Y_{ik} and Z_{ik}.

$$X_{ik} = \begin{array}{cc} & \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{pmatrix} (0.9,0.1) & (0.8,0.15) & (0.9,0.08) & (0.88,0.11) \\ (0.9,0.1) & (0.8,0.16) & (0.8,0.12) & (0.85,0.14) \\ (0.9,0.08) & (0.8,0.15) & (0.9,0.08) & (0.87,0.11) \\ (0.73,0.1) & (0.8,0.15) & (0.83,0.09) & (0.85,0.11) \end{pmatrix} \end{array}$$

$$Y_{ik} = \begin{array}{cc} & \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{pmatrix} (0.73,0.22) & (0.68,0.27) & (0.79,0.15) & (0.81,0.18) \\ (0.73,0.19) & (0.75,0.25) & (0.77,0.22) & (0.8,0.18) \\ (0.73,0.19) & (0.75,0.2) & (0.79,0.19) & (0.79,0.18) \\ (0.73,0.19) & (0.7,0.19) & (0.79,0.19) & (0.79,0.18) \end{pmatrix} \end{array}$$

$$Z_{ik} = \begin{array}{cc} & \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} & \begin{pmatrix} (0.815,-0.06) & (0.74,-0.035) & (0.845,-0.035) & (0.845,-0.025) \\ (0.815,-0.045) & (0.775,-0.045) & (0.825,-0.05) & (0.825,-0.015) \\ (0.815,-0.055) & (0.775,-0.025) & (0.845,-0.055) & (0.835,-0.025) \\ (0.815,-0.045) & (0.75,-0.02) & (0.81,-0.05) & (0.82,-0.03) \end{pmatrix} \end{array}$$

Step 5: By comparing the elements of first column we conclude that patient P_1 is affected by chicken pox. Similarly we conclude that P_2 is affected by small pox, P_3 is affected by mumps and P_2 is affected by measles while P_4 is also affected by measles.

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Optimality Conditions for Fuzzy Non-Linear Equality Constrained Minimization Problems

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ABSTRACT

In this paper, an optimality conditions for fuzzy non-linear equality constrained minimization problems are discussed. Here the cost coefficients and constrained coefficients are represented by a triangular fuzzy number. Some numerical illustrations are discussed by using these optimality conditions.

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Keywords: fuzzy non-linear equality constrained minimization problem, triangular fuzzy number, triangular fuzzy matrix.

1. INTRODUCTION

Many authors considered different types of the fuzzy non-linear programming problems and proposed several approaches by solving these problems. R. E. Bellman and L. A. Zadeh [5] have introduced the decision making in a fuzzy environment. Hsien-Chung Wu [10] has presented an $(\alpha, \beta)-$ optimal solution concept in fuzzy optimization problems and also he [11] discussed the optimality conditions for optimization problems with fuzzy-valued objective functions. V. D. Pathak and U. M. Pirzada [19] have introduced the necessary and sufficient optimality conditions for nonlinear fuzzy optimization problem. R. Saranya and Palanivel Kaliyaperumal [20] have presented fuzzy nonlinear programming problem for inequality constraints with alpha optimal solution in terms of trapezoidal membership functions.

Here, either an objective function or constraints or both of them are non-linear and all the variables are considered as fuzzy variables. Here, the necessary and sufficient optimality conditions are based on the concept of partial differentiability of the Lagrangian function are discussed. Numerical examples based on these optimality conditions are given.

2. PRELIMINARIES

2.1. Fuzzy non-linear programming problem:

It refers to an optimization problem in which the variables are continuous variables and the problem is of the following general form:

$$\text{Minimize } \theta(\tilde{x})$$
$$\text{Subject to } h_i(\tilde{x}) = 0,\ i = 1, 2, \dots, m.$$
$$g_p(\tilde{x}) \geq 0,\ p = 1, 2, \dots, t.$$

where $\theta(\tilde{x}), h_i(\tilde{x}), g_p(\tilde{x})$ are all real valued continuous functions of

$\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in \mathrm{R}^n$.

2.3. Fuzzy local minimum:

Consider a fuzzy non-linear programming problem in which a function $\theta(\tilde{x})$ is required to be optimized subject to some constraints on the variables $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$. Let $\tilde{K}$ denote the set of fuzzy feasible solutions for this problem. For this problem a fuzzy feasible solution $\tilde{x} \in \tilde{K}$ is said to be a local minimum, if there exists an $\in < 0$ such that $\theta(\tilde{x}) \geq \theta(\tilde{x})$ for all $\tilde{x} \in \tilde{K} \cap \{\tilde{x} : \| \tilde{x} - \tilde{x} < \in \}$.

2.4. Fuzzy set

A fuzzy set $\tilde{P}$ is defined by $\tilde{P} = \{(s, \mu_P(s)) : s \in P, \mu_P(s) \in [0, 1]\}$. In the pair $(s, \mu_P(s))$, the first element s belong to the classical set P, the second element $\mu_P(s)$ belong to the interval [0, 1], called membership function.

2.5. Fuzzy number

The notion of fuzzy numbers was introduced by Dubois D and Prade H [26]. A fuzzy subset $\tilde{P}$ of the real line R with membership function $\mu_{\tilde{P}} : R \to [0, 1]$ is called a fuzzy number if

(i) A fuzzy set $\tilde{P}$ is normal.

(ii) $\tilde{P}$ is fuzzy convex, that is

$$\mu_{\tilde{P}}[\lambda_{s_1} + (1 - \lambda)s_2] \geq \mu_{\tilde{P}}(s_1) \wedge \mu_{\tilde{P}}(s_2),\ s_1, s_2 \in R,\ \forall \lambda \in [0, 1].$$

(iii) $\mu_{\tilde{P}}$ is upper continuous and

Supp $\tilde{P}$ is bounded, where supp $\tilde{P} = \{s \in R : \mu_{\tilde{P}}(s) > 0\}$.

2.6. Triangular Fuzzy Number

The triangular fuzzy number can be denoted as $\tilde{P} = (p_1, p_2, p_3)$, where p_2 is the central value, $\mu_{\tilde{P}}(p_2) = 1$, such that $p_1 < p_2 < p_3$ are defined in R.

The α-cut of a triangular fuzzy number is,

$$\tilde{P}_\alpha = [(p_2 - p_1)\alpha + p_1, -(p_3 - p_2)\alpha + p_3].$$

The membership function $\mu_{\tilde{P}}(s)$ is given by,

$$\mu_{\tilde{P}}(s) = \begin{cases} 0 & \text{for } s < p_1 \\ \dfrac{s - p_1}{p_2 - p_1} & \text{for } p_1 \le s \le p_2 \\ \dfrac{p_3 - s}{p_3 - p_2} & \text{for } p_2 \le s \le p_3 \\ 0 & \text{for } s > p_3. \end{cases}$$

2.7. Operations of Triangular Fuzzy Number using Function Principle

Let $\tilde{P} = (p_1, p_2, p_3)$ and $\tilde{Q} = (q_1, q_2, q_3)$ be two triangular fuzzy numbers.

Then

(i) The addition of $\tilde{P}$ and $\tilde{Q}$ is

$$\tilde{P} + \tilde{Q} = (p_1 + q_1, p_2 + q_2, p_3 + q_3)$$

where $p_1, p_2, p_3, q_1, q_2, q_3$ are real numbers.

(ii) The product of $\tilde{P}$ and $\tilde{Q}$ is

$$\tilde{P} \times \tilde{Q} = (c_1, c_2, c_3), \text{ where } T = \{p_1 q_1, p_2 q_2, p_3 q_3\}$$

where $c_1 = \min \{T\}$, $c_2 = p_2 q_2$, $c_3 = \max \{T\}$.

If $p_1, p_2, p_3, q_1, q_2, q_3$ are all non-zero positive real numbers, then

$$\tilde{P} \times \tilde{Q} = (p_1 q_1, p_2 q_2, p_3 q_3).$$

(iii) $-\tilde{Q} = (-q_3, -q_2, -q_1)$.

Then the subtraction of $\tilde{Q}$ from $\tilde{P}$ is

$$\tilde{P} - \tilde{Q} = (p_1 - q_3, p_2 - q_2, p_3 - q_1),$$

where $p_1, p_2, p_3, q_1, q_2, q_3$ are real numbers.

(iv) $\dfrac{1}{\tilde{Q}} = \tilde{Q}^{-1} = \left(\dfrac{1}{q_3}, \dfrac{1}{q_2}, \dfrac{1}{q_1}\right)$

$\dfrac{\tilde{P}}{\tilde{Q}} = (c_1, c_2, c_3)$, where $T = \left(\dfrac{p_1}{q_3}, \dfrac{p_2}{q_2}, \dfrac{p_3}{q_1}\right)$ where

$$c_1 = \min \{T\}, \; c_2 = \frac{p_2}{q_2}, \; c_3 = \max \{T\}.$$

If $p_1, p_2, p_3, q_1, q_2, q_3$ are all non-zero positive real numbers, then

$$\frac{\tilde{P}}{\tilde{Q}} = \left(\frac{p_1}{q_3}, \frac{p_2}{q_2}, \frac{p_3}{q_1}\right).$$

(v) Let $\alpha \in R$, then $\alpha\tilde{P} = (\alpha p_1, \alpha p_2, \alpha p_3)$, if $\alpha \ge 0$

$= (\alpha p_3, \alpha p_2, \alpha p_1)$ if $\alpha < 0$.

2.8. Triangular Fuzzy Matrix

A triangular fuzzy matrix of order $m \times n$ is defined as $P = (\tilde{p}_{ij})_{m \times n}$,

where $\tilde{p}_{ij} = (p_{ij1}, p_{ij2}, p_{ij3})$ is the ij^{th} element of P.

2.9. Operations on Triangular Fuzzy Matrices

Let $S = (\tilde{s}_{ij})$ and $T = (\tilde{t}_{ij})$ be two triangular fuzzy matrices of same order. Then

(i) $S + T = (\tilde{s}_{ij} + \tilde{t}_{ij})$

(ii) $S - T = (\tilde{s}_{ij} - \tilde{t}_{ij})$

(iii) For $S = (\tilde{s}_{ij})_{m \times n}$ and $T = (\tilde{t}_{ij})_{n \times k}$ then $ST = (\tilde{c}_{ij})_{m \times k}$

where $\tilde{c}_{ij} = \sum_{p=1}^{n} \tilde{s}_{ip} \cdot \tilde{t}_{pj}$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$.

(iv) $S^T = (\tilde{s}_{ji})$

(v) $KS = (K\tilde{s}_{ij})$ where K is scalar.

2.10. Positive semi-definite fuzzy matrix

A fuzzy square matrix $\tilde{A} = (\tilde{a}_{ij})$ of order n, whether it is symmetric or not, is said to be a positive semi definite fuzzy matrix if $\tilde{x}^T \tilde{A} \tilde{x} \geq 0$ for all $\tilde{x} \in \mathrm{R}^n$.

2.11. Positive definite fuzzy matrix

A fuzzy square matrix $\tilde{A} = (\tilde{a}_{ij})$ of order n, whether it is symmetric or not, is said to be a positive definite fuzzy matrix if $\tilde{x}^T \tilde{A} \tilde{x} > 0$ for all $\tilde{x} \neq 0$.

3. OPTIMALITY CONDITIONS FOR FUZZY NON-LINEAR EQUALITY CONSTRAINED MINIMIZATION PROBLEMS

Consider the fuzzy non-linear programming problem,

minimize $\theta(\tilde{x})$

Subject to $h_i(\tilde{x}) = 0$, $i = 1$ to m (3.1)

where $\theta(\tilde{x})$, $h_i(\tilde{x})$ are all real valued continuously differentiable functions defined on R^n. Let $h(\tilde{x}) = (h_1(\tilde{x}), \dots, h_m(\tilde{x}))^T$. The set of fuzzy feasible solutions is a surface in R^n, and it is smooth if each $h_i(\tilde{x})$ is a smooth function (i.e., continuously differentiable). If $\bar{\tilde{x}}$, is a fuzzy feasible point, when some of the $h_i(\tilde{x})$ are nonlinear, there may be no fuzzy feasible direction at $\tilde{x}$. In order to retain fuzzy feasibility while moving from $\bar{\tilde{x}}$, one has to follow a nonlinear curve through $\bar{\tilde{x}}$ which lies on the fuzzy feasible surface.

A curve in R^n is the locus of a point $\tilde{x}(\lambda) = (\tilde{x}_j(\lambda))$ where each $\tilde{x}_j(\lambda)$ is a real valued function of the real parameter λ, as the parameter varies over some interval of the real line.

The curve $\tilde{x}(\lambda) = (\tilde{x}_j(\lambda))$ is said to be differentiable at λ if $\frac{d\tilde{x}_j(\lambda)}{d\lambda}$ exists for all j, and twice differentiable if $\frac{d^2\tilde{x}_j(\lambda)}{d\lambda^2}$ exists for all j. The curve $\tilde{x}(\lambda)$ is said to pass through the point $\bar{\tilde{x}}$ if $\bar{\tilde{x}} = \tilde{x}(\bar{\lambda})$ for some $\bar{\lambda}$.

If the curve $\tilde{x}(\lambda)$ defined over $a < \lambda < b$ is differentiable at $\bar{\lambda}$, $a < \bar{\lambda} < b$, then the line $\{\tilde{x} = \tilde{x}(\bar{\lambda}) + \delta \frac{d\tilde{x}}{d\lambda}(\bar{\lambda}) : \delta$ real$\}$ is the tangent line to the curve at the point $\tilde{x}(\bar{\lambda})$ on it.

The tangent plane at a fuzzy feasible point $\bar{\tilde{x}}$ to (3.1) is defined to be the set of all directions $\left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)_{\lambda=0}$, where $\tilde{x}(\lambda)$ is a differential curve in the fuzzy feasible region with $\tilde{x}(0) = \bar{\tilde{x}}$.

Theorem 3.1. *If* $\bar{\tilde{x}}$ *is a fuzzy regular point for* (3.1), *the tangent plane for* (3.1) *at* $\bar{\tilde{x}}$ *is* $\{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$.

Proof. Let $\tilde{x}(\alpha)$ be a differentiable curve lying in the fuzzy feasible region for α lying in an interval around zero, with $\tilde{x}(0) = \bar{\tilde{x}}$ and $\frac{d\tilde{x}(0)}{d\alpha} = \tilde{y}$. So $h(\tilde{x}(\alpha)) = 0$ for all values of α lying in an interval around zero, and hence $\left(\frac{dh(\tilde{x}(\alpha))}{d\alpha}\right)_{\alpha=0} = 0$, that is $(\nabla h(\bar{\tilde{x}}))\tilde{y} = 0$. This implies that the tangent plane is a subset of $\{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$.

Suppose $\tilde{y} \in \{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$ and $\tilde{y} \neq 0$.

Define new variables $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)^T$. Consider the following system of m equations in $m+1$ variables $\tilde{u}_1, \ldots, \tilde{u}_m, \alpha$.

$$\tilde{g}_i(\tilde{u}, \alpha) = \tilde{h}_i(\bar{\tilde{x}} + \alpha\tilde{y} + (\nabla h(\bar{\tilde{x}}))^T \tilde{u}) = 0, \; i = 1 \text{ to } m. \tag{3.2}$$

$\tilde{g}(0, 0) = 0$ and the Jacobian matrix of $\tilde{g}(\tilde{u}, \alpha)$ with respect to $\tilde{u}$ is non-singular at $\tilde{u} = 0$, $\alpha = 0$ (since $\bar{\tilde{x}}$ is a regular point of (3.1)). So by applying the implicit function theorem on (3.2), we can express $\tilde{u}$ as a differentiable function of α, say $\tilde{u}(\alpha)$, in an interval around $\alpha = 0$, and that (3.2) holds as an identity in this interval when $\tilde{u}$ in (3.2) is replaced by $\tilde{u}(\alpha)$, and that $\tilde{u}(0) = 0$, and $\frac{d\tilde{u}(0)}{d\alpha}$ is obtained by solving

$\left(\frac{d}{d\alpha} h(\bar{\tilde{x}} + \alpha\tilde{y} + (\nabla h(\bar{\tilde{x}}))^T \tilde{u}(\alpha)\right)_{\alpha=0} = 0$ which leads to $\frac{d}{d\alpha}\tilde{u}(0) = 0$ since $\nabla h(\bar{\tilde{x}})$ has rank m.

So if we define $\tilde{x}(\alpha) = \bar{\tilde{x}} + \alpha\tilde{y} + (\nabla h(\bar{\tilde{x}}))^T \tilde{u}(\alpha)$.

This defines a differentiable curve lying in the feasible region for (3.1) for values of α in an interval around $\alpha = 0$, and that $\frac{d\tilde{x}}{d\alpha}\tilde{u} = \tilde{y}$, which implies that $\tilde{y}$ is in the tangent plane for (3.1) at $\bar{\tilde{x}}$.

We will now derive optimality conditions for (3.1) by using theorem (3.1).

If $\bar{\tilde{x}}$ is a fuzzy feasible regular point for (3.1), and it is a local minimum, clearly along every differentiable curve $\tilde{x}(\alpha)$ lying in the fuzzy feasible region for (3.1) for values of α in an interval around $\alpha = 0$, satisfying $\bar{\tilde{x}}(0) = \bar{\tilde{x}}$; $\alpha = 0$ must be a local minimum for $\theta(\bar{\tilde{x}})$ on this curve. That is, for the problem of minimizing $\theta(\bar{\tilde{x}}(\alpha))$ over this interval for α, $\alpha = 0$ must be a local minimum. Since $\alpha = 0$ is an interior point of this interval this implies that $\frac{d\theta}{d\alpha}(\tilde{x}(0))$ must be zero. Applying this to all such curves and using theorem 3.1 we conclude that $(\nabla\theta(\tilde{x}))\tilde{y} = 0$ for all $\tilde{y}$ satisfying $(\nabla h(\bar{\tilde{x}}))\tilde{y} = 0$.

There must exist $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)$ such that $(\nabla\theta(\bar{\tilde{x}})) - \sum_{i=1}^{m} \bar{\mu}_i \nabla h_i(\bar{\tilde{x}}) = 0$

and by feasibility $h_i(\bar{\tilde{x}}) = 0$ (3.3) the conditions (3.3) are the first order necessary optimality conditions for (3.1), the vector $\tilde{\mu}$ is the vector of Lagrange multipliers. (3.1) is a system of $(n + m)$ equations in $(n + m)$ unknowns (including $\bar{\tilde{x}}$ and $\bar{\tilde{\mu}}$) and it may be possible to solve (3.3) using algorithms for solving nonlinear equations. If we define the Lagrangian for (3.1) to be $L(\tilde{x}, \mu) = \theta(\tilde{x}) - \mu h(\tilde{x})$ where $\mu = (\mu_1, \mu_m)$, $h(\tilde{x}) = (h(\tilde{x}), h_m(\tilde{x}))^T$, (3.3) becomes: $(\bar{\tilde{x}}, \bar{\tilde{\mu}})$ satisfies

$$h(\tilde{x}) = 0$$

$$\nabla_{\tilde{x}} L(\tilde{x}, \mu) = 0. \tag{3.4}$$

We will now derive the second order necessary optimality conditions for (3.1). Suppose the functions $\theta(\tilde{x}), h_i(\tilde{x})$ are all twice continuously differentiable. Let $\bar{\tilde{x}}$ be a fuzzy feasible solution for (3.1) which is a regular point. If $\bar{\tilde{x}}$ is a local minimum for (3.1), by the first order necessary optimality conditions (3.4), there must exist a row vector of Lagrange multipliers, $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)$ such that $\nabla_{\tilde{x}} L(\tilde{x}, \bar{\mu}) = 0$, where $L(\tilde{x}, \bar{\mu}) = \theta(\tilde{x}) - \bar{\mu} h(\tilde{x})$ is the Lagrangian. Since $\bar{\tilde{x}}$ is a fuzzy regular point, the tangent plane to (3.1) at $\bar{\tilde{x}}$ is $T = \{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$. Suppose there exists a $\bar{\tilde{y}} \in T$ satisfying $\bar{\tilde{y}}^T H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))\bar{\tilde{y}} < 0$. Since $\bar{\tilde{y}} \in T$, and all the functions are twice continuously differentiable, there exists a twice differentiable curve $\tilde{x}(\lambda)$ through $\bar{\tilde{x}}$ lying in the feasible region (i.e., $\tilde{x}(0) = \bar{\tilde{x}}$, and the curve is defined in an interval of λ with 0 as an interior point, with $h(\tilde{x}(\lambda)) = 0$ for all λ in this interval), such that $\left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)_{\lambda=0} = \bar{\tilde{y}}$.

Now, $\frac{d}{d\lambda} L(\tilde{x}, \lambda, \bar{\mu}) = (\nabla_{\tilde{x}} L(\tilde{x}(\lambda), \bar{\tilde{\mu}}))\left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)$

$$\frac{d^2}{d\lambda^2} L(\tilde{x}(\lambda), \bar{\mu}) = \left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)^T H_{\tilde{x}}(L(x(\lambda), \bar{\mu}))\frac{d\tilde{x}(\lambda)}{d\lambda} + (\nabla_{\tilde{x}} L(\tilde{x}(\lambda), \bar{\mu}))\left(\frac{d^2\tilde{x}(\lambda)}{d\lambda^2}\right),$$

where $\nabla_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu})), H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))$ are the fuzzy row vector of partial derivatives with respect to $\tilde{x}$ and the Hessian matrix with respect to $\tilde{x}$ of $L(\bar{\tilde{x}}, \bar{\mu})$ at $\tilde{x} = \bar{\tilde{x}}$ respectively. At $\lambda = 0$, we have $\nabla_{\tilde{x}} L(\tilde{x}(0), \bar{\mu}) = \nabla_{\tilde{x}} L(\bar{\tilde{x}}, \bar{\mu}) = 0$ by the first order necessary optimality conditions.

So, from the above

$$\left(\frac{d}{d\lambda} L(\tilde{x}(\lambda), \bar{\mu})\right)_{\lambda=0} = 0$$

$$\left(\frac{d^2}{d\lambda^2} L(\tilde{x}(\lambda), \bar{\mu})\right)_{\lambda=0} = \bar{\tilde{y}}^T H_{\tilde{x}}(L\bar{\tilde{x}}, \bar{\mu}))\bar{\tilde{y}}.$$

Using these in a Taylor series expansion for $f(\lambda) = L(\tilde{x}(\lambda), \overline{\mu})$ up to second order around $\lambda = 0$ leads to $f(\lambda) = L(\tilde{x}(\lambda), \overline{\mu}) = L(\overline{\tilde{x}}, \overline{\mu}) + \frac{\lambda^2}{2} \tilde{\tilde{y}}^T$ $H_{\tilde{x}}(L(\overline{\tilde{x}}, \overline{\mu}))\tilde{\tilde{y}} + o(\lambda)$ where $o(\lambda) = 0$, since $\lim_{\lambda \to 0} \frac{o(\lambda)}{\lambda^2} = 0$.

Since $h(\tilde{x}(\lambda)) = 0$ for every point on the curve, we have $f(\lambda) = L(\tilde{x}(\lambda), \overline{\mu}) = \theta(\tilde{x}(\lambda))$ for all λ in the interval of λ on which the curve is defined. So in the neighbourhood of $\lambda = 0$ on the curve we have from the above

$$\frac{2(\theta(\tilde{x}(\lambda))) - \theta(\overline{\tilde{x}})}{\lambda^2} = \frac{2(f(\lambda) - f(0))}{\lambda^2} = \tilde{\tilde{y}}^T H_{\tilde{x}}(L(\overline{\tilde{\tilde{x}}}, \overline{\mu}))\tilde{\tilde{y}} + \frac{2(o(\lambda))}{\lambda^2}$$

and since $\tilde{\tilde{y}}^T H_{\tilde{x}}(L(\overline{\tilde{x}}, \overline{\mu}))\tilde{\tilde{y}} < 0$ and $\lim_{\lambda \to 0} \frac{o(\lambda)}{\lambda^2} = 0$, for all λ sufficiently small $\theta(\tilde{x}(\lambda)) - \theta(\overline{\tilde{x}}) < 0$. For all these λ, $\tilde{x}(\lambda)$ is a point on the curve in the fuzzy feasible region in the neighbourhood of $\tilde{x}$, and this is a contradiction to the fact that $\overline{\tilde{x}}$, is a fuzzy local minimum for (3.1).

In fact it can be verified that $\tilde{\tilde{y}}^T H_{\tilde{x}}(L(L\overline{\tilde{x}}, \overline{\mu}))\tilde{\tilde{y}} = \left(\frac{d^2 f(\lambda)}{d\lambda^2}\right)_{\lambda=0}$ and if this quantity is < 0, $\lambda = 0$ cannot be a local minimum for the one variable minimization problem of minimizing $f(\lambda) = \theta(\tilde{x}(\lambda))$ over λ; or equivalently, that $\overline{\tilde{x}} = \tilde{x}(0)$ is not a local minimum for $\theta(\tilde{x})$ along the curve $\tilde{x}(\lambda)$.

These facts imply that if $\theta(\tilde{x})$, $h_i(\tilde{x})$ are all twice continuously differentiable, and $\overline{\tilde{x}}$ is a fuzzy regular point which is a fuzzy feasible solution and a fuzzy local minimum for (3.1), there must exist a Lagrange multiplier fuzzy vector $\overline{\mu}$ such that the following conditions hold.

$$h(\overline{\tilde{x}}) = 0$$
$$\nabla_{\tilde{x}} L(\overline{\tilde{x}}, \overline{\mu}) = \nabla \theta(\overline{\tilde{x}}) - \overline{\mu} \nabla h(\overline{\tilde{x}}) = 0$$
$$\tilde{y}^T H_{\tilde{x}}(L(\overline{\tilde{x}}, \overline{\mu})) \geq 0 \text{ for all } \tilde{y} \in T = \{\tilde{y} : (\nabla h((\overline{\tilde{x}})))\tilde{y} = 0\}, \qquad (3.5)$$

i.e., $H_{\tilde{x}}(L((\overline{\tilde{x}}), \overline{\mu}))$ is *PSD* on the subspace *T*.

These are the second order necessary optimality conditions for a fuzzy regular feasible point $\overline{\tilde{x}}$ to be a fuzzy local minimum for (3.1).

Theorem 3.2 (Sufficient optimality condition for (3.1)). *Suppose $\theta(\tilde{x})$, $h_i(\tilde{x})$, $i = 1$ to m are all twice continuously differentiable functions, and $\overline{\tilde{x}}$ is a fuzzy feasible point such that there exists a Lagrange multiplier vector $\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_m)$ which together satisfy*

$$h(\overline{\tilde{x}}) = 0$$

$$\nabla \theta(\overline{\tilde{x}}) - \overline{\mu} \nabla h(\overline{\tilde{x}}) = 0$$

$$\tilde{y}^T H_{\tilde{x}}(L(\overline{\tilde{x}}, \overline{\mu}))\tilde{y} > 0 \textit{ for all } \tilde{y} \in \{\tilde{y} : \nabla h((\overline{\tilde{x}}))\tilde{y} = 0\},\ \tilde{y} \neq 0 \qquad (3.6)$$

where $L(\tilde{x}, \overline{\mu}) = \theta(\tilde{x}) - \overline{\mu} h(\tilde{x})$ *is the Lagrangian for* (3.1).

Then $\overline{\tilde{x}}$ is a local minimum for (3.1).

Proof. Suppose $\overline{\tilde{x}}$ is not a local minimum for (3.1). There must exist a sequence of distinct fuzzy feasible points $\{\tilde{x}^r : r = 1, 2, \ldots\}$ converging to $\overline{\tilde{x}}$ such that $\theta(\tilde{x}^r) < \theta(\overline{\tilde{x}})$ for all r.

Let $\delta_r = |\overline{\tilde{x}} - \tilde{x}^r|$, $\tilde{y}^r = (\overline{\tilde{x}} - \tilde{x}^r)/\delta_r$.

Then $\| \tilde{y}^r \| = 1$ for all r and $\tilde{x}^r = \overline{\tilde{x}} + \delta_r \tilde{y}^r$.

Thus $\delta_r \to 0^+$ as $r \to \infty$.

Since the sequence of points $\{\tilde{y}^r : r = 1, 2, \ldots, \}$ all lie on the surface of the unit sphere in R^n, a compact set, the sequence has atleast one limit point.

Let $\tilde{\tilde{y}}$ be a limit point of $\{\tilde{y}^r : r = 1, 2, \ldots, \}$.

There must exist a subsequence of $\{\tilde{y}^r : r = 1, 2, \ldots, \}$ which converges to $\tilde{\tilde{y}}$, eliminate all points other than those in this subsequence, and for simplicity call the remaining sequence by the same notation $\{\tilde{y}^r : r = 1, 2, \ldots, \}$.

So now we have a sequence of points $\tilde{x}^r = \overline{\tilde{x}} + \delta_r \tilde{y}^r$ all of them fuzzy feasible, such that $\| \tilde{y}^r \| = 1$ for all r, $\tilde{y}^r \to \tilde{\tilde{y}}$ and $\delta_r \to 0$ as $r \to \infty$.

By feasibility $h(\overline{\tilde{x}} + \delta_r \tilde{y}^r) = 0$ for all r, and by the differentiability of $h(\tilde{x})$ we have

$$0 = h(\overline{\tilde{x}} + \delta_r \tilde{y}^r) = h(\overline{\tilde{x}}) + \delta_r \nabla h(\overline{\tilde{x}})\tilde{y}^r + o(\delta_r)$$
$$= \delta_r \nabla h(\overline{\tilde{x}})\tilde{y}^r + o(\delta_r).$$

Dividing by $\delta_r > 0$, and taking the limit as $r \to \infty$ we see that $\nabla h(\overline{\tilde{x}})\tilde{\tilde{y}} = 0$.

Since $L(\tilde{x}, \overline{\mu})$ is a twice continuously differentiable function in $\tilde{x}$, applying Taylor's theorem to it, we conclude that for each r, there exists a

$0 \leq \alpha_r \leq \delta_r$ such that $L(\bar{\tilde{x}} + \delta_r \tilde{y}^r, \bar{\mu}) = L(\bar{\tilde{x}} \; \bar{\mu}) + \delta_r \nabla_{\tilde{x}} L(\bar{\tilde{x}}, \bar{\mu}) \tilde{y}^r$ $+ (1/2)\delta_r^2 (\tilde{y}^r)^T H_{\tilde{x}}(\bar{\tilde{x}} + \alpha_r \tilde{y}^r, \bar{\mu}))\tilde{y}^r.$

From the fact that $\bar{\tilde{x}} + \delta_r \tilde{y}^r = \tilde{x}^r$ and $\bar{\tilde{x}}$ are feasible, we have $L(\tilde{x}^r, \bar{\mu}) = \theta(\tilde{x}^r)$ and $L(\bar{\tilde{x}}, \bar{\mu}) = \theta(\bar{\tilde{x}})$. Also, from (3.6), $\nabla_{\tilde{x}} L(\bar{\tilde{x}}, \bar{\mu}) = 0$. So, from the above equation, we have

$$\theta(\tilde{x}^r) - \theta(\bar{\tilde{x}}) = (1/2)\delta_r^2 (\tilde{y}^r)^T H_{\tilde{x}}(L(\bar{\tilde{x}} + \alpha_r \tilde{y}^r, \bar{\mu}))\tilde{y}^r. \tag{3.7}$$

Since $0 \leq \alpha_r \leq \delta_r$ and $\delta_r \to 0$ as $r \to \infty$, and by continuity, $H_{\tilde{x}}(L(\bar{\tilde{x}} + \alpha_r \bar{\tilde{y}}_r, \bar{\mu}))$ converges to $H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))$ as $r \to \infty$. Since $\tilde{y}^r \to \tilde{y}$ as $r \to \infty$, and $\nabla h(\bar{\tilde{x}})\bar{\tilde{y}} = 0$, from the last condition in (3.6) and contiuity we conclude that when r is sufficiently large, the right-hand side of (3.7) is ≥ 0, while the left-hand side is < 0, a contradiction. So, $\bar{\tilde{x}}$ must be a fuzzy local minimum for (3.1).

Thus, (3.7) provides a sufficient condition for a fuzzy feasible point $\bar{\tilde{x}}$ to be a fuzzy local minimum for (3.1).

4. NUMERICAL EXAMPLE

Example 4.1. Consider the problem minimize

$$(-1.25, -1, -0.75)\tilde{s}_1 + (-1.25, -1, -0.75)\tilde{s}_2 \text{ subject to}$$

$$(0.75, 1, 1.25)\tilde{s}_1^2 + (0.75, 1, 1.25)\tilde{s}_2^2 + (-8.25, -8, -7.75) = 0.$$

Solution

Given constraint is

$$(0.75, 1, 1.25)\tilde{s}_1^2 + (0.75, 1, 1.25)\tilde{s}_2^2 + (-8.25, -8, -7.75) = 0. \tag{4.1}$$

The Lagrangian is

$$L(\tilde{s}, \lambda) = (-1.25, -1, -0.75)\tilde{s}_1 + (-1.25, -1, -0.75)\tilde{s}_2$$
$$- \lambda[(0.75, 1, 1.25)\tilde{s}_1^2 + (0.75, 1, 1.25)\tilde{s}_2^2 + (-8.25, -8, -7.75)].$$

The first order necessary optimality conditions are

$$\frac{\partial L(\tilde{s}, \lambda)}{\partial \tilde{s}} = [(-1.25, -1, 0.75) - 2(0.75, 1, 1.25)\lambda \tilde{s}_1, (-1.25, -1, -0.75)$$
$$-2(0.75, 1, 1.25)\lambda \tilde{s}_2 = 0.$$

$$\Rightarrow s_1 = \frac{1}{\lambda}(-0.5, -0.5, -0.5) \tag{4.2}$$

$$\tilde{s}_2 = \frac{1}{\lambda}(-0.5, -0.5, -0.5) \tag{4.3}$$

Using (4.2) and (4.3) in (4.1), we get

$$\lambda^2 = (0.05, 0.06, 0.08)$$
$$\lambda = \mp(0.22, 0.24, 0.28)$$
$$\lambda = -(0.22, 0.24, 0.28).$$
$$\tilde{s}_1 = (1.78, 2.08, 2.27)$$
$$\tilde{s}_2 = (1.78, 2.08, 2.27)$$

$$\text{Therefore, } \bar{\tilde{s}} = \begin{bmatrix} (1.78, 2.08, 2.27) \\ (1-78, 2.08, 2.27) \end{bmatrix}$$

$$\bar{\lambda} = -(0.22, 0.24, 0.28)$$

$$\frac{\partial L(\tilde{s})}{\partial \tilde{s}} = [(-1.25, -1, -0.75) + (0.33, 0.48, 0.7)\tilde{s}_1, (-1.25, -1, -0.75) + (0.33, 0.48, 0.7)\tilde{s}_1].$$

The Hessian of the Lagrangian is

$$H_{\tilde{s}}(L(\bar{\tilde{s}}, \bar{\lambda})) = \begin{bmatrix} (0.33, 0.48, 0.7) & (0, 0, 0) \\ (0, 0, 0) & (0.33, 0.48, 0.7) \end{bmatrix}$$

$$\begin{aligned} \tilde{t}^T H_{\tilde{s}}(L(\bar{\tilde{s}}, \bar{\lambda}))\tilde{t} &= [\tilde{t}_1, \tilde{t}_2]\begin{bmatrix} (0.33, 0.48, 0.7) & (0, 0, 0) \\ (0, 0, 0) & (0.33, 0.48, 0.7) \end{bmatrix}\begin{bmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{bmatrix} \\ &= (0.33, 0.48, 0.7)\tilde{t}_1^2 + (0.33, 0.48, 0.7)\tilde{t}_2^2 \\ &= (0.33, 0.48, 0.7)[\tilde{t}_1^2 + \tilde{t}_2^2] \\ &> \tilde{0} \end{aligned}$$

$H_{\tilde{s}}(L(\bar{\tilde{s}}, \bar{\lambda}))$ is *PD*.

Hence $\bar{\tilde{s}}$ satisfies the sufficient condition for being a fuzzy local minimum in this problem.

5. CONCLUSION

In this paper, the fuzzy nonlinear equality constrained minimization problem is defined and the optimality conditions for this problem are stated. Some examples are discussed based on these optimality conditions.

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